

Thesis for the degree of Master of Science in Theoretical Physics

# Boundary Dynamics of Three-Dimensional Asymptotically Anti-de Sitter Space-Times

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# Chapter 1

## Introduction

Nature knows of four fundamental forces - electromagnetic, weak, strong, and gravitational. The electrical and magnetic interactions were initially considered as two distinct phenomena, but in the late 19th century, Maxwell discovered a way to view them as two sides of the same coin, and the term electromagnetism was born. The weak and strong interactions describe phenomena at very short distance scales. Gravity, on the other hand, works at such large length scales that we all sense its effect. Since Einstein introduced his theory of general relativity in 1916, it has singled itself out as an accurate description of the gravitational interaction and the curvature of space-time. Gravity is the one fundamental force that has been most difficult to incorporate into a consistent picture of nature. The other three forces have already been combined into a theory called the standard model in the 1970s. The problem with also incorporating gravity is that the standard model is a quantum field theory, but the gravitational field does not allow itself to be quantized in a straightforward way.

Out of the four fundamental interactions, we will mainly be concerned with gravity. However, we will not be working in the  $3 + 1$  (three spatial, one time) dimensions of our physical world, but rather in  $2 + 1$  dimensions, where gravity is simplified because it does not possess any local propagating degrees of freedom. What makes the theory interesting after all are global properties [1], and a phenomenon we call ‘boundary dynamics.’ Based on a work by J. D. Brown and M. Henneaux (1986) [2], we will look at the group of asymptotic symmetries of a class of asymptotically anti-de Sitter space-times. Anti-de Sitter space-time is the maximally symmetric solution of the Einstein equations with constant negative curvature and no matter sources. ‘Asymptotically anti-de Sitter’ means that we require the space-time to behave like anti-de Sitter space-time near spatial infinity. The presence of the boundary partially breaks diffeomorphism invariance, causing only certain transformations (asymptotic symmetries) to remain as symmetries.

Brown and Henneaux noticed that the asymptotic symmetry algebra is special in two respects. For one, the  $\text{AdS}_3$  isometry group is extended to an infinite-dimensional group at the boundary, namely the group of conformal transformations of the plane. Since the conformal group has only a finite number of generators in higher dimensions, this is a phenomenon specific to the  $2 + 1$ -dimensional theory. Besides the fact that the asymptotic symmetries form an infinite-dimensional group, Brown and Henneaux noticed that their Poisson bracket algebra is *centrally extended*. They also calculated the associated central charge. The fact that the central extension arises already at the classical level makes it even more special, because central extensions are best known as a quantum effect. In this thesis it will become clear that central extensions may very well arise classically.

The asymptotic symmetry group is generated in the Hamiltonian formalism by two copies of the Virasoro algebra, suggesting a dual interpretation in terms of a conformal field theory on the  $(1 + 1)$ -dimensional boundary at spatial infinity. It was shown in [3] that this conformal field theory is Liouville theory. A supersymmetric generalization of this was presented in [4, 5] for the case  $\mathcal{N} = 1$  and in [6] for extended supergravity. These results are reminiscent of the AdS/CFT correspondence which was conjectured by Maldacena in 1998 [7], and concerns an analogy between  $(d - 1)$ -dimensional supersymmetric conformal field theory and black branes of which the near-horizon geometry is  $\text{AdS}_d$  times a compact manifold.

The conformal field theory interpretation has led to another interesting result. There is a well-known formula due to Cardy [8] relating the central charge of two-dimensional conformal field theory to the density of high-energy states in the theory. The logarithm of the density of states gives an expression for the entropy. Strominger [9] has shown that, using the central charge of asymptotically  $\text{AdS}_3$  space-times, Cardy's formula yields exactly the Bekenstein-Hawking [10, 11] entropy of a black hole solution discovered by Bañados, Teitelboim and Zanelli (BTZ) in 1992 [12]. The BTZ black hole is a three-dimensional solution of Einstein's equations with negative cosmological constant and a point source at the origin. Far away from the source, the solution behaves like anti-de Sitter space, and the BTZ black hole metric is asymptotically anti-de Sitter in the sense of [2]. Generalizations of Strominger's approach to different solutions have been suggested for instance in [14].

We have performed our analysis in the metric formulation, following Brown and Henneaux. An alternative would have been the Chern-Simons formulation of  $(2 + 1)$ -dimensional gravity [15], which is a gauge theory in terms of a one-form field  $A_i$  taking values in the space-time isometry group. The specific case of (asymptotically)  $\text{AdS}_3$  was studied in the Chern-Simons formulation in [16] and [17]. The derivations of the conformal field theories on the boundary in [3] and [6] also make use of this formulation, of which the supersymmetric generalization was given in [18].

This thesis is organized as follows. In Chapter 2, some preliminaries are given on isometries and conformal symmetries, and we become familiar with the Virasoro algebra. Two examples of classical central charges are discussed. Chapter 3 contains an introduction to the Hamiltonian formulation of gauge theories in the context of Maxwell theory. The knowledge gained in Chapter 3 is applied to general relativity in Chapter 4. The Hamiltonian is shown to acquire a surface term due to the presence of the boundary. The geometrical properties of anti-de Sitter space and the BTZ black hole are the subject of Chapter 5. The main part of the discussion follows in Chapter 6, which contains the calculation of the central charge in the asymptotic symmetry algebra of asymptotically  $\text{AdS}_3$  space-times. After some preliminaries on the AdS/CFT correspondence and Chern-Simons theory, the derivation of the boundary conformal field theory [3] is summarized in Chapter 7. Finally, Strominger's entropy calculation is presented in Chapter 8, along with some comments on this approach.

Throughout, our metrics will have signature  $(- + \dots +)$ , where the first entry denotes the time component.

## Chapter 2

# Isometries and Conformal Symmetry

In this chapter we introduce the notions of isometries and conformal symmetry. We motivate their introduction by considering the symmetry properties of the scalar field theory defined by the action

$$S = \int \sqrt{-g(x)} g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) d^d x. \quad (2.1)$$

Section 2.3 deals with the central extension of the conformal algebra in two dimensions, also known as the Virasoro algebra. In section 2.4, we discuss a few examples of theories that are centrally extended at the classical level.

### 2.1 Coordinate Transformation

First consider the theory (2.1) in arbitrary dimension  $d$ . We define the Lagrangian density  $\mathcal{L}(g, \phi)$  to be the entire integrand, including the factor  $\sqrt{-g(x)}$ . The equation of motion is

$$D^\mu \partial_\mu \phi = 0. \quad (2.2)$$

The metric is not treated as a dynamical field here, because our action does not contain an Einstein-Hilbert term. Since on scalar fields the covariant derivative is simply the partial derivative, we can still expect there to be some equivalence between initial and transformed Lagrangians under general diffeomorphisms. We will look exactly what happens under the infinitesimal transformation  $x^\mu \rightarrow x^\mu - \xi^\mu(x)$ . Performing an active transformation, such that we describe the new fields in terms of the old coordinates  $x^\mu$ , we have

$$\begin{aligned} \delta \sqrt{-g} &= \frac{1}{2} \delta g^\mu_\mu \sqrt{-g} = \left( \frac{1}{2} g^{\mu\nu} \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\rho \xi^\rho \right) \sqrt{-g}, \\ \delta \partial_\mu \phi &= \partial_\mu (\xi^\nu \partial_\nu \phi) = \partial_\mu \xi^\nu \partial_\nu \phi + \xi^\nu \partial_\nu \partial_\mu \phi, \\ \delta g^{\mu\nu} &= \xi^\rho \partial_\rho g^{\mu\nu} - g^{\rho\nu} \partial_\rho \xi^\mu - g^{\rho\mu} \partial_\rho \xi^\nu. \end{aligned} \quad (2.3)$$

Notice that by  $\delta g^\mu_\mu$  we mean the trace of  $\delta g_{\mu\nu}$ , not the variation of the trace of  $g_{\mu\nu}$ , which would be  $\xi^\rho \partial_\rho g^\mu_\mu$ . It can be verified that under these transformations,

$$\delta \mathcal{L} = \partial_\nu (\xi^\nu \mathcal{L}), \quad (2.4)$$

so that we obtain the equivalence statement that under infinitesimal general coordinate transformations,  $\mathcal{L}(g, \phi)$  goes into a new Lagrangian  $\mathcal{L}(g', \phi')$  which has the same equations of motion but in terms of different fields. This is not a true invariance, as the background metric has changed in the process. Because the metric is not a dynamical field in our example, we have to choose a specific metric beforehand instead of determining it from the equations of motion. Therefore, the next thing to do is to look at invariances of the metric (isometries), generated by so-called Killing vectors. This will lead to a transformed Lagrangian  $\mathcal{L}(g, \phi')$  describing the same theory as  $\mathcal{L}(g, \phi)$ , only the scalar field having changed.

Under the infinitesimal transformation  $x^\mu \rightarrow x^\mu - \xi^\mu(x)$ , the metric changes by the Lie derivative (as we already used in contravariant form in (2.3)), which can be rewritten as a sum of covariant derivatives of the deformation vector:

$$\begin{aligned}
g'_{\mu\nu}(x) - g_{\mu\nu}(x) &= \mathcal{L}_\xi g_{\mu\nu} \\
&= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho \\
&= \partial_\mu (g_{\nu\rho} \xi^\rho) + \partial_\nu (g_{\mu\rho} \xi^\rho) - 2\Gamma_{\mu\nu}^\beta \xi_\beta \\
&= \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^\beta \xi_\beta + \partial_\nu \xi_\mu - \Gamma_{\nu\mu}^\beta \xi_\beta \\
&= D_\mu \xi_\nu + D_\nu \xi_\mu
\end{aligned} \tag{2.5}$$

where we assumed the connection to be the Christoffel connection

$$\Gamma_{\mu\nu}^\beta = \frac{1}{2} g^{\beta\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}), \tag{2.6}$$

which makes sure that  $D_\rho g_{\mu\nu} = 0$ . From (2.5) it follows that Killing vectors  $\xi^\mu$  obey

$$D_\mu \xi_\nu + D_\nu \xi_\mu = 0, \tag{2.7}$$

an equation with at most  $\frac{1}{2}d(d+1)$  linearly independent solutions. For instance, flat Minkowski space is a maximally symmetric space, its Killing vectors corresponding to the  $\frac{1}{2}d(d+1)$  generators of the Poincaré group ( $\frac{1}{2}d(d-1)$  Lorentz transformations and  $d$  translations).

The variation of the Lagrangian is proportional to the field equations under any transformation of  $\phi$ . Since the action is invariant under diffeomorphisms, the part of the variation proportional to  $\delta g_{\mu\nu}$  is therefore independently proportional to the field equations for any diffeomorphism  $\xi^\mu$ . If we introduce the energy-momentum, or stress, tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}, \tag{2.8}$$

we can denote this part of the variation by

$$\delta S_g = - \int \frac{1}{2} \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} d^d x. \tag{2.9}$$

Because  $\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$ , and  $T^{\mu\nu}$  is symmetric in its indices, we can integrate by parts to get

$$\delta S_g = \int \sqrt{-g} \xi_\nu D_\mu T^{\mu\nu} d^d x \tag{2.10}$$



Since  $\xi_\nu$  is arbitrary, this shows that  $T^{\mu\nu}$  is covariantly conserved,  $D_\mu T^{\mu\nu} = 0$ , in accordance with Noether's theorem that every continuous symmetry of the action brings along a conserved current.

In our present example, using  $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}$ , it can be verified that (2.9) gives

$$T_{\mu\nu} = g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi g_{\mu\nu} - 2\partial_\mu \phi \partial_\nu \phi. \quad (2.11)$$

It can be checked that  $T_{\mu\nu}$  is covariantly conserved if the scalar field obeys its equation of motion.

The conserved (divergenceless) current  $T_{\mu\nu}$  implies the existence of  $d$  conserved charges

$$Q_\mu = \int T^0{}_\mu d^{d-1}x \quad (2.12)$$

where the index 0 stands for time and we integrate over the spatial directions. Indeed, we have

$$\begin{aligned} \partial_0 Q_\mu &= \int \partial_0 T^0{}_\mu d^{d-1}x \\ &= - \int \partial_i T^i{}_\mu d^{d-1}x, \end{aligned} \quad (2.13)$$

which vanishes by virtue of Gauss's theorem assuming  $T^i{}_\mu$  vanishes at the boundaries (a Latin index denotes spatial components).

## 2.2 Conformal Symmetry

In general, a theory can have more symmetries than only those generated by the Killing vectors. We could for instance try to rescale the metric by a local factor (a *Weyl* rescaling),  $g_{\mu\nu}(x) \rightarrow f(x)g_{\mu\nu}(x)$ . One example for which this is a symmetry is four-dimensional Maxwell theory, for which the Lagrangian in terms of the antisymmetric field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  reads

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\mu} F_{\beta\nu}. \quad (2.14)$$

With the Lagrangian written in this form, it is easy to see that in four dimensions the theory is invariant under Weyl rescalings, since  $g^{\mu\nu}(x) \rightarrow \frac{1}{f(x)}g^{\mu\nu}(x)$ , and in four dimensions  $\sqrt{-g}$  transforms with  $f^2(x)$ .

It is sometimes also possible to achieve a rescaling of the metric through diffeomorphisms  $x^\mu \rightarrow x^\mu - \xi^\mu(x)$ , a *conformal* transformation. Four-dimensional source-free Maxwell theory in flat space-time is conformally invariant. As soon as a source term  $A_\mu J^\mu$  is added to the Lagrangian, conformal invariance is lost. Besides four-dimensional Maxwell theory, there are many more examples of conformally invariant theories. Nonlinear sigma models in flat space-time and with conical target space can be made scale invariant in any dimension by adding appropriate improvement terms [19]. Mass terms generally break scale invariance.

A field  $\Psi$  is called conformally invariant with conformal weight or conformal dimension  $w$  if  $\Psi$  is a solution of the field equations with metric  $g_{\mu\nu}$ , and  $\Omega^w \Psi$

is a solution with metric  $\Omega^2 g_{\mu\nu}$ . Fields obtained by functional differentiation of a conformally invariant action with respect to the metric are conformally invariant. In particular, the stress tensor of a conformally invariant theory is conformally invariant. For tensor fields, the conformal weight depends on the index positions.

The vectors achieving a rescaling of the metric obey the equation

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \frac{2}{d} g_{\mu\nu} D_\rho \xi^\rho. \quad (2.15)$$

Vectors obeying (2.15) are called *conformal* Killing vectors. They form a group, just like the Killing vectors. From the right-hand side it can be seen that they indeed rescale the metric by a local factor, since the left-hand side is the variation of  $g_{\mu\nu}$  under subtraction of  $\xi^\mu$  from the coordinates. Whereas the Killing equations require  $D_\mu \xi_\nu + D_\nu \xi_\mu$  to be traceless, the conformal Killing equations lead to no such restriction. Therefore there are generally more conformal Killing vectors than there are Killing vectors. Indeed, in more than two dimensions, the conformal Killing vectors of flat  $d$ -dimensional Minkowski spacetime generate the so-called conformal group  $SO(d, 2)$ , consisting of  $d$  translations,  $\frac{1}{2}d(d-1)$  Lorentz transformations, one dilatation and  $d$  so-called special conformal transformations. The total conformal group therefore has  $\frac{1}{2}(d+1)(d+2)$  generators, whereas the Killing vectors only corresponded to the  $\frac{1}{2}d(d+1)$  generators of the Poincaré group. The transformations of the conformal group in more than two dimensions and their infinitesimal forms are summarized in Table 2.1.

Operation	Finite Form	Generator
translations	$x'^\mu = x^\mu + \lambda^\mu$	$P_\mu = \partial_\mu$
Lorentz transformations	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$
dilatations	$x'^\mu = A x^\mu$	$D = -x^\mu \partial_\mu$
special conformal transformations	$x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b - x^2 b^2}$	$K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$

Table 2.1: The conformal group for  $d > 2$ .

We can define the new generators

$$\begin{aligned}
G_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu), \\
G_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu), \\
G_{-1,0} &= D, \\
G_{\mu\nu} &= L_{\mu\nu},
\end{aligned} \quad (2.16)$$

where we introduced two extra indices, so that Latin indices take on the values  $-1, 0, \dots, d$ , while Greek indices run over  $1, 2, \dots, d$ . In terms of the generators

$G_{ab}$ ,  $SO(d, 2)$  is described by the algebra

$$[G_{ab}, G_{cd}] = \eta_{ac}G_{bd} + \eta_{db}G_{ac} - \eta_{ad}G_{bc} - \eta_{bc}G_{ad} \quad (2.17)$$

with  $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, \dots, 1)$ .

For  $d \geq 3$ , the conformal group in  $d$  dimensions corresponds to the anti-de Sitter group in  $d + 1$  dimensions. The anti-de Sitter group is the group of isometries of anti-de Sitter space. We will come back to this in Chapter 5.

We already know that the number of linearly independent conformal Killing vectors does not exceed  $\frac{1}{2}(d+1)(d+2)$  for dimensions greater than two. However, in two dimensions, the conformal group is extended to an infinite-dimensional group. To see this, we turn to holomorphic coordinates  $z, \bar{z}$ , for which the metric has only the components  $g_{z\bar{z}} = g_{\bar{z}z} = 1$ <sup>1</sup>. This can be done because any two-dimensional metric is conformally flat. With the given form of the metric, it can be checked that each of the vectors

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \quad (2.18)$$

$n \in \mathbb{Z}$ , satisfies the conformal Killing equations (2.15). Hence there is an infinite number of conformal Killing vectors in  $d = 2$ . The vectors  $l_n$  obey the so-called loop, or Witt, algebra

$$[l_m, l_n] = (m - n)l_{m+n}, \quad (2.19)$$

and similarly for  $\bar{l}_n$ , whereas the  $l_n$  and  $\bar{l}_m$  commute among each other. The term loop algebra derives from the fact that if one sets  $z = e^{i\phi}$ , (2.19) describes the algebra of vector fields on a circle in the complex plane. The six vectors  $l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1$  generate a subgroup  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})/Z_2$  isomorphic to  $SO(2, 2)$ . This group is often written as the product of a left-hand and a right-hand group  $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ .

Now consider the theory (2.1) in  $d = 2$ . Because in two dimensions  $g$ , the determinant of  $g_{\mu\nu}$ , transforms with the inverse square of  $g^{\mu\nu}$ , we see that the Lagrangian is invariant under local rescalings of the metric. So two is a special number of dimensions in this respect.

Associated with the conformal invariance is again a conserved Noether current  $J_\mu$ , which now looks like

$$J_\mu = T_{\mu\nu}\xi^\nu, \quad (2.20)$$

where  $T_{\mu\nu}$  is the energy-momentum tensor specified before, and  $\xi^\nu$  a conformal Killing vector.  $J_\mu$  is covariantly conserved by virtue of

$$\begin{aligned} D_\mu J^\mu &= \xi^\nu D_\mu T^\mu_\nu + T^\mu_\nu D_\mu \xi^\nu \\ &= \frac{1}{2}T^{\mu\nu}(D_\mu \xi_\nu + D_\nu \xi_\mu) \\ &= \frac{1}{d}T^\mu_\mu D_\rho \xi^\rho \\ &= \left(1 - \frac{2}{d}\right)\partial^\mu \phi \partial_\mu \phi D_\rho \xi^\rho. \end{aligned} \quad (2.21)$$

---

<sup>1</sup>The ‘holomorphic’ coordinates will be complex in the Euclidean case, but real in  $(1+1)$ -dimensional Minkowski space. In the latter case,  $z = \sigma + \tau$ ,  $\bar{z} = \sigma - \tau$ , with  $\sigma$  the spatial and  $\tau$  the time coordinate, and  $z$  and  $\bar{z}$  are more commonly referred to as lightcone coordinates.

The last expression obviously vanishes in the case  $d = 2$ . The third line immediately shows that conformal invariance implies the vanishing of the trace of the energy-momentum tensor,  $T^\mu_\mu = 0$ .

According to a well-established theorem, the stress tensor classically generates transformations of the canonical variables through the Poisson bracket. Let us consider this in two dimensions. In holomorphic coordinates  $T_{z\bar{z}}(z, \bar{z})$  vanishes, and  $\bar{T}_{zz}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z})$ , so that it has become common practice to denote  $T_{zz}(z)$  simply by  $T(z)$ . The latter can then be expanded in terms of an infinite number of operators  $L_n$  according to

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad (2.22)$$

and similarly for  $\bar{T}(\bar{z})$  in terms of  $\bar{L}_n$ . The components  $L_n$  of the energy-momentum tensor  $T_{\mu\nu}$  will generate symmetries of the two-dimensional Lagrangian, just like  $T_{\mu\nu}$  itself, and hence will form an algebra. On the classical configuration space, the  $L_n$  defined by (2.22) turn out to obey the same algebra as the vectors  $l_n$ , namely

$$\{L_m, L_n\} = (m - n)L_{m+n} \quad (2.23)$$

So the symmetry group of the scalar field theory (2.1) in two dimensions is generated by two copies of the loop algebra discussed above. In some theories a modified version of this algebra arises, and this is discussed in the next section.

## 2.3 Central Extension

We are still looking at the infinite-dimensional group of conformal transformations that arises in  $d = 2$ . We mentioned that the algebra (2.19) may be modified in some cases. The modified algebra is called a *central extension* of the loop algebra.

When quantizing the two-dimensional theory, the components  $L_n, \bar{L}_n$  of the energy-momentum tensor become operators on the quantum mechanical Hilbert space. They can then be expanded in terms of products of raising and lowering operators. After commuting two different  $L_m, L_n$ , these raising and lowering operators will be mixed up, and since they do not commute among each other, the result will not immediately be recognizable as another  $L_{n'}$  or  $\bar{L}_{n'}$ . In order to close the algebra after all, we need to give a prescription for the order in which to put the raising and lowering operators in the expansion. This is called normal ordering. Conventionally, all raising operators are written to the left of the lowering operators. This leads to an additional term in the algebra which commutes with all the generators and is called ‘central’ for that reason. However, normal ordering is not the only way in which to obtain a central charge. In Chapter 6 we will come across a central extension of the Virasoro algebra associated with asymptotic isometries of a class of asymptotically  $AdS_3$  spacetimes which is not quantum in nature. A classical central charge was first described by Gervais and Neveu in the context of Liouville theory [22]. More on the topic of classical central charges is explored in section 2.4.

The central extension of the loop algebra, the *Virasoro* algebra, is best known from string theory. For string theory, two is a special number of dimensions, since it is the dimension of the worldsheet swept out by the strings, and string theory has a  $d = 2$  conformal invariance.

Let us have a look at the relation between the quantum mechanical, centrally extended loop algebra and its classical limit. In order to pass to quantum mechanics, we first replace the Poisson bracket by  $\frac{1}{i\hbar}$  times the commutator:

$$[L_m, L_n] = i\hbar(m - n)L_{m+n}. \quad (2.24)$$

Defining  $L'_m \equiv \frac{L_m}{i\hbar}$ , we have

$$[L'_m, L'_n] = (m - n)L'_{m+n}. \quad (2.25)$$

Performing the normal ordering leads to (we denote normal ordered operators by a script letter)

$$[\mathcal{L}'_m, \mathcal{L}'_n] = (m - n)\mathcal{L}'_{m+n} + \text{central term}. \quad (2.26)$$

Such a centrally extended algebra has so-called *projective representations*. We can formally take the classical limit by going back to  $\mathcal{L}_m \equiv i\hbar\mathcal{L}'_m$ , obtaining

$$[\mathcal{L}_m, \mathcal{L}_n] = i\hbar\left((m - n)\mathcal{L}_{m+n} + i\hbar * \text{central term}\right), \quad (2.27)$$

subsequently replacing the commutator by  $i\hbar$  times the Poisson bracket, and taking the limit  $\hbar \rightarrow 0$ . The central term then drops out.

By making use of the Jacobi identity and through addition of constants to the generators it can be shown that the central extension can always be written in the form (henceforth dropping primes)

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m - n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}(m^3 - m)\delta_{m+n,0}, \\ [\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0, \end{aligned} \quad (2.28)$$

where  $c$  is an undetermined central charge (the factor  $\frac{1}{12}$  is according to convention). The commutation relations (2.28) are those of two copies of the Virasoro algebra with central charge  $c$ . The notation is somewhat misleading, since, whereas the  $\mathcal{L}_n$  can be interpreted as generators, the central term is just a constant of which it is not clear what it generates. The generators  $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1, \bar{\mathcal{L}}_{-1}, \bar{\mathcal{L}}_0, \bar{\mathcal{L}}_1$  form a subalgebra isomorphic to  $so(2, 2)$  without central extension, which is obvious with the  $m$ -dependence in (2.28).

The central charge also shows up in the transformation law of the two-dimensional stress tensor. Using the components of the stress tensor as generators of transformations through the Lie bracket, we can derive the infinitesimal variation of, in this case, the stress tensor itself. Of course this works not only for  $T(z)$ , but for any function on the Hilbert space. Afterwards, we also give the finite form of the variation.

Just like the  $l_n$ ,  $\mathcal{L}_n$  generates the transformation  $z \rightarrow z - \epsilon z^{n+1}$ , and (leaving out the infinitesimal parameter  $\epsilon$ ) we have

$$\delta_{z^{n+1}} T(z) = [\mathcal{L}_n, T(z)]. \quad (2.29)$$

Again,  $T(z)$  can be expanded in terms of the Virasoro operators as

$$T(z) = \sum_{m \in \mathbb{Z}} \mathcal{L}_m z^{-m-2}, \quad (2.30)$$

and we get

$$\begin{aligned} \delta_{z^{n+1}} T(z) &= [\mathcal{L}_n, \sum_{m \in \mathbb{Z}} \mathcal{L}_m z^{-m-2}] \\ &= \sum_{m \in \mathbb{Z}} z^{-m-2} ((n-m)\mathcal{L}_{m+n} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}) \\ &= (z^{n+1}\partial_z + 2\partial_z z^{n+1}) \sum_{m \in \mathbb{Z}} \mathcal{L}_m z^{-m-2} + \frac{c}{12} \partial_z^3 z^{n+1}. \end{aligned} \quad (2.31)$$

The same can be done for general  $n$ , so that we arrive at (from now on denoting  $\partial_z$  simply by  $\partial$ )

$$\delta_\xi T(z) = (\xi\partial + 2\partial\xi)T(z) + \frac{c}{12}\partial^3\xi. \quad (2.32)$$

The first part is tensorial (i.e. it is according to the tensor transformation law  $T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} T^{\rho\sigma}(x)$ ), in contrast to the second part, which features the same central charge as appears in the central extension of the Virasoro algebra.

We will now look at the finite form of the transformation (2.32). If  $f(z) = z + \xi(z) + O(\xi^2)$ , the transformation law (2.32) is the infinitesimal version of

$$T(z) = (\partial f)^2 T(f(z)) + \frac{c}{12} \{f, z\} \quad (2.33)$$

where the Schwarzian derivative  $\{f, z\}$  is defined by

$$\{f, z\} = \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}. \quad (2.34)$$

This can be seen by calculating (2.33) up to first order in  $\xi$ .

## 2.4 Classical Central Charges

In the above discussion we have treated the central term as a quantum effect, which it is in most cases. However, we have mentioned that central charges may perfectly well arise already at the classical level. This section introduces the concept of classical central charges by means of two examples. One is a scalar field theory in  $(1+1)$  dimensions with constant external electric field, in which the center of the symmetry algebra actually generates a transformation. The other is Liouville theory, where the central term only shows up at the level of the Poisson brackets, and does not act as a generator. We will see that

there is a difference in the mechanisms by which these two central terms arise. The possibility for a central charge that does not generate a transformation derives from an ambiguity in the canonical generators. The central charge in the first example, on the other hand, is shown to arise due to a partially broken symmetry that is restored by additional symmetries.

Our first example of a classical central extension was described by E. Karat in [20]. It concerns a charged scalar field theory in flat  $(1+1)$ -dimensional space with a constant external electric field. The Lagrangian is

$$\mathcal{L} = -(D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi \quad (2.35)$$

with

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi + ie A_\mu \phi \\ A_\mu &= -\frac{1}{2} \epsilon_{\mu\nu} x^\nu F, \end{aligned} \quad (2.36)$$

and  $\epsilon^{10} = -\epsilon_{10} = 1$ . The electric field  $E$  then becomes

$$E = \partial_0 A_1 - \partial_1 A_0 = F, \quad (2.37)$$

and it does not point in any direction. Due to the lack of a dynamical term for  $A_\mu$ , the vector potential does not transform, and the action is not manifestly invariant under translations. If we pretended the  $A_\mu$ -field to transform after all, the total symmetry group would be a product of the Poincaré group, consisting of Lorentz transformations and translations

$$\begin{aligned} \phi &\rightarrow \phi + t^\mu \partial_\mu \phi \\ A_\mu &\rightarrow A_\mu - \frac{1}{2} \epsilon_{\mu\nu} F t^\nu \\ \delta_T \mathcal{L} &= \delta_{T_\phi} \mathcal{L} + \delta_{T_A} \mathcal{L} \\ &= t^\mu \partial_\mu \mathcal{L}, \end{aligned} \quad (2.38)$$

and gauge transformations

$$\begin{aligned} \phi &\rightarrow e^{ie\Lambda} \phi \\ A_\mu &\rightarrow A_\mu - \partial_\mu \Lambda \\ \delta_g \mathcal{L} &= \delta_{g_\phi} \mathcal{L} + \delta_{g_A} \mathcal{L} \\ &= 0. \end{aligned} \quad (2.39)$$

However, the change in  $A_\mu$  under translations is just a gauge transformation

$$\begin{aligned} \delta_T A_\mu &= -\frac{1}{2} \epsilon_{\mu\nu} F t^\nu \\ &= \partial_\mu \left( -\frac{1}{2} \epsilon_{\nu\rho} x^\rho t^\nu \right), \end{aligned} \quad (2.40)$$

as we could have guessed, because the electric field is itself translation invariant. From (2.39), we then see that we can ‘translate’ the transformation of  $A_\mu$  under

translations into an infinitesimal gauge transformation on  $\phi$ , obtaining a so-called *covariant translation*:

$$\delta\phi = t^\mu(\partial_\mu + \frac{1}{2}ie\epsilon_{\mu\nu}x^\nu F)\phi. \quad (2.41)$$

Defining the transformation of  $\phi$  under translations to be (2.41), we recover translation invariance of the action when  $A_\mu$  does not transform,

$$\delta_T \mathcal{L} = t^\mu \partial_\mu [(-D^\nu \phi)^* D_\nu \phi - m^2 \phi^* \phi]. \quad (2.42)$$

However, the covariant translations do not commute like regular translations do,

$$\begin{aligned} [\delta_L, \delta_T^\mu] \phi &= \epsilon^{\mu\nu} (\delta_T)_\nu \phi \\ [\delta_T^\mu, \delta_T^\nu] \phi &= ie\epsilon^{\mu\nu} F \phi. \end{aligned} \quad (2.43)$$

For simplicity, we have left out the translation parameters. The operator that multiplies with  $ieF$  is a central element, since it commutes with the other symmetry operations. The same algebra is obeyed by the corresponding Noether charges, but the central term  $ie\epsilon^{\mu\nu} F$  already shows up before introducing Poisson brackets. The above example gives a good idea of how central charges may show up already at the classical level.

In our next example, the center does not act as a generator. The possibility for such a central term in a Poisson bracket algebra can be explained as follows, with an argument from [21]. Suppose the symmetries of the action close according to

$$[\delta_a, \delta_b] \phi = f_{ab}^c \delta_c \phi, \quad (2.44)$$

and we have a set of Noether charges generating these symmetries,

$$\delta_a \phi = \{Q_a, \phi\}, \quad (2.45)$$

where  $\{, \}$  denotes the Poisson bracket,

$$\{A, B\} \equiv \sum_n \frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial B}{\partial q_n} \frac{\partial A}{\partial p_n} \quad (2.46)$$

for canonical positions  $q_n$  and canonical momenta  $p_n$ . Then the condition (2.44) amounts to

$$\{Q_a, Q_b\} = f_{ab}^c Q_c + k \Delta_{ab} \quad (2.47)$$

where the only restrictions are that  $k \Delta_{ab}$  should have vanishing Poisson brackets with the fields, and  $\Delta_{ab}$  is antisymmetric in its indices. This is because

$$\{f_{ab}^c Q_c + k \Delta_{ab}, \phi\} = \{f_{ab}^c Q_c, \phi\} = f_{ab}^c \delta_c \phi. \quad (2.48)$$

Of course,  $k \Delta_{ab}$  may, in particular, vanish. Thus, the possibility for a central charge in the Poisson bracket algebra that does not act as a generator derives from an ambiguity in the Noether charges.



An example that illustrates this is two-dimensional Liouville theory. The Hamiltonian reads [22]

$$H = \int \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_\sigma \phi)^2 + e^\phi - 2\partial_\sigma^2 \phi \right\} d\sigma, \quad (2.49)$$

where the ranges of the coordinates are  $\tau \in (-\infty, \infty)$ ,  $\sigma \in [0, \pi]$ . The canonical momentum is  $\pi = \partial_\tau \phi$ . The last term  $-2\partial_\sigma^2 \phi$  is a boundary term which is added to make sure the variation of the Hamiltonian is well-defined. That is, together with the boundary conditions

$$\begin{aligned} \partial_\sigma \phi &= -\sqrt{2} \rho e^{\phi/2} \quad \text{at } \sigma = 0, \\ \partial_\sigma \phi &= \sqrt{2} \rho e^{\phi/2} \quad \text{at } \sigma = \pi, \end{aligned} \quad (2.50)$$

where  $\rho$  is a scale parameter, the variation of the action with respect to  $\phi(\tau, \sigma)$  vanishes neatly at the endpoints of  $\sigma$ . The corresponding equation of motion is the Liouville equation

$$\phi_{\tau\tau} - \phi_{\sigma\sigma} + e^\phi = 0. \quad (2.51)$$

This theory is conformally invariant. The Noether charges can be expanded according to

$$\begin{aligned} L_0 &= - \int \left\{ \left( \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_\sigma \phi)^2 + e^\phi - 2\partial_\sigma^2 \phi \right) \right\} d\sigma - 2\pi \\ L_m &= - \int \left\{ \left( \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_\sigma \phi)^2 + e^\phi - 2\partial_\sigma^2 \phi \right) \cos m\sigma \right. \\ &\quad \left. + i(\pi \partial_\sigma \phi - 2\partial_\sigma^2 \pi) \sin m\sigma \right\} d\sigma, \end{aligned} \quad (2.52)$$

and their Poisson bracket algebra is the Virasoro algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n} - 4\pi i(m^3 - m)\delta_{m+n,0}. \quad (2.53)$$

The central extension is claimed [22] to be a result of the presence of a boundary term in the Hamiltonian. As opposed to the previous example, the central charge of this theory is not a generator on the classical configuration space. Chapter 6 will reveal that two copies of precisely such an algebra are encountered when describing the asymptotic symmetries of a class of AdS<sub>3</sub>-like metrics.

## Chapter 3

# Constraints: An Example from Maxwell Theory

General relativity is sometimes regarded as the gauge theory of diffeomorphisms, where the role of the gauge field is played by the metric. Gauge invariance is reflected in the Hamiltonian formalism by relations between the canonical variables, so-called *constraints*. If the canonical positions are  $q_n$  and the canonical momenta  $p_n$ , they can be expressed by  $\phi_m(q_n, p_n) = 0$ . Some of the  $\phi_m$  generate gauge transformations. Constraints also arise in the Hamiltonian formulation of general relativity. In order to get an intuitive feeling, we first discuss the relatively simple example of free Maxwell theory, before turning to the more complicated case of general relativity in the next chapter.

As mentioned in section 2.2, the free Maxwell Lagrangian is

$$L(A_\mu, \partial_0 A_\mu) = -\frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} d^3x, \quad (3.1)$$

with the field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and  $A_\mu$  the photon field. The index 0 stands for the time parameter that needs to be integrated over in order to obtain the full action. For simplicity, we consider the theory in flat Minkowski space-time. The components of the field strength tensor are

$$F_{0i} = E_i, \quad F_{ij} = -\epsilon_{ijk} B^k, \quad (3.2)$$

where  $\epsilon_{ijk}$  is the fully antisymmetric Levi-Civita symbol,  $E_i$  is the electric field and  $B_i$  the magnetic field. Conversely, we then have  $B^k = \epsilon^{ijk} \partial_i A_j$ , or in vector notation  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

The equations of motion

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu A_\nu)} = \frac{\delta L}{\delta A_\nu} \quad (3.3)$$

read

$$\partial_\mu F^{\mu\nu} = 0. \quad (3.4)$$

The remaining Maxwell equations follow from the Bianchi identity

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0. \quad (3.5)$$

According to the rules of field theory, the canonical ‘position’ becomes the field  $A_\mu$ , and the corresponding ‘momentum’ is

$$\pi^\mu \equiv \frac{\delta L}{\delta(\partial_0 A_\mu)} = F^{\mu 0}. \quad (3.6)$$

We see that  $\pi^i$  is just the electric field. It is already clear that the canonical momentum  $\pi^0$  vanishes due to the antisymmetry of  $F^{\mu\nu}$ . In other words,  $\pi^0$  is one of the constraints of the theory, on a par with the identity  $\vec{\nabla} \cdot \vec{B} = 0$  (telling us there are no magnetic monopoles). Because we can immediately see it vanishes, it is called a *primary* constraint. So-called *secondary* constraints may follow from the requirement that the primary constraints be consistent under time evolution. One can repeat this procedure until no new constraints arise.

Performing the Legendre transformation from the Lagrangian to the Hamiltonian yields a Hamiltonian that is only defined up to the addition of constraints, the *total* Hamiltonian ( $\mathcal{L}$  denotes the Lagrangian density)

$$\begin{aligned} H_T(A_\mu, \pi^\mu) &= \int \left\{ (\dot{A}_\mu) \pi^\mu - \mathcal{L} \right\} d^3x \\ &= \int \left\{ \dot{A}_0 \pi^0 + \dot{A}_i F^{i0} - \frac{1}{2} F^{i0} F_{i0} + \frac{1}{4} F^{ij} F_{ij} \right\} d^3x \\ &= \int \left\{ \frac{1}{2} \pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} - A_0 (\partial_i \pi^i) + \dot{A}_0 \pi^0 \right\} d^3x. \end{aligned} \quad (3.7)$$

Notice that the transformation from the Lagrangian to the Hamiltonian formulation is invertible only if we treat  $\dot{A}_0$  as a Lagrange multiplier for the constraint  $\pi^0 = 0$ . The Hamiltonian (3.7) further differs from the more familiar

$$H = \int \frac{1}{2} (\vec{E}^2 + \vec{B}^2) d^3x \quad (3.8)$$

by the term  $-A_0(\vec{\nabla} \cdot \vec{E})$ . We can thus expect  $-\partial_i \pi^i$  to be another constraint, representing Gauss’s law. We will now derive how this comes about on formal grounds.

As mentioned above,  $\pi^0 = 0$  should be preserved in time in order for this to be a consistent constraint. When calculating the time derivative of  $\pi^0$  we cannot simply set it to zero from the outset. The time evolution belonging to (3.7) is

$$\dot{f} = \{f, H\} \quad (3.9)$$

for any function  $f(A_\mu, \pi^\mu)$  that does not depend explicitly on time. Choosing

$f = \pi^0$ , we have

$$\begin{aligned}
\{\pi^0(\vec{x}), H\} &= \int \left( \left\{ \pi^0(\vec{x}), \frac{1}{2} \pi^i \pi_i(\vec{x}') \right\} + \left\{ \pi^0(\vec{x}), \frac{1}{4} F^{ij} F_{ij}(\vec{x}') \right\} \right. \\
&\quad \left. - \left\{ \pi^0(\vec{x}), A_0(\partial_i \pi^i)(\vec{x}') \right\} + \left\{ \pi^0(\vec{x}), \dot{A}_0 \pi^0(\vec{x}') \right\} \right) d\vec{x}' \\
&= \iint \left\{ \frac{\delta A_0(\partial_i \pi^i)(\vec{x}')}{\delta A_\rho(\vec{x}'')} \frac{\delta \pi^0(\vec{x})}{\delta \pi^\rho(\vec{x}'')} \right\} d\vec{x}' d\vec{x}'' \\
&= -\partial_i \pi^i(\vec{x}) \\
&\equiv 0.
\end{aligned} \tag{3.10}$$

We have thus found a secondary constraint which is recognized as the source-free Gauss law  $\vec{\nabla} \cdot \vec{E} = 0$ . No further constraints arise, since the bracket of  $\partial_i \pi^i$  with the Hamiltonian is zero.

The surface defined by  $\pi^0 = \partial_i \pi^i = 0$  is a smooth submanifold of the configuration (or phase) space called the constraint surface. In particular, the physical content of a function on phase space is represented by the value of the function on this surface. We call functions that may differ off the constraint surface, but coincide on it, *weakly* equal. Observe that there may be many physically distinct solutions satisfying the constraints. Each of these can be evolved with the Hamiltonian according to (3.9). There exists a similar initial value formulation of general relativity.

In the derivation of both equations of motion and constraints, no restriction whatsoever has been placed on the Lagrange multipliers  $A_0$  and  $\dot{A}_0$ . Thus, the time evolution (3.9) contains arbitrary functions. Since, given a Hamiltonian, we assume the physical content of a state to fully determine its physical content at another moment in time, the difference resulting from a difference in  $A_0$  and  $\dot{A}_0$  should be unobservable; in other words, it should be a gauge transformation. This is in agreement with the so-called Dirac conjecture, which states that all *first-class* constraints - constraints which have vanishing Poisson brackets with all other constraints - are generators of gauge transformations.

To see what the Gauss constraint generates, it is convenient to integrate with a parameter  $\lambda(\vec{x}')$ :

$$\begin{aligned}
&\{A_k(\vec{x}), \int \partial_i \pi^i(\vec{x}') \lambda(\vec{x}') d^3 x'\} \\
&= \int \left\{ \frac{\delta A_k(\vec{x})}{\delta A_j(\vec{x}'')} \frac{\delta(\int \partial_i \pi^i(\vec{x}') \lambda(\vec{x}') d^3 x')}{\delta \pi^j(\vec{x}'')} - \frac{\delta(\int \partial_i \pi^i(\vec{x}') \lambda(\vec{x}') d^3 x')}{\delta A_j(\vec{x}'')} \frac{\delta A_k(\vec{x})}{\delta \pi^j(\vec{x}'')} \right\} d^3 x'' \\
&= \int \left\{ \delta^k_j \delta(\vec{x} - \vec{x}'') \frac{\delta(-\int \pi^i(\vec{x}') \partial_i \lambda(\vec{x}') d^3 x')}{\delta \pi^j(\vec{x}'')} \right\} d^3 x'' \\
&= \partial_k \lambda(\vec{x}).
\end{aligned} \tag{3.11}$$

This is recognized as the  $U(1)$  gauge transformation

$$\delta A_i = \partial_i \lambda, \tag{3.12}$$

and we see that this first-class constraint indeed generates gauge transformations. All in all, the arbitrariness of  $A_0$  together with the gauge freedom (3.12)

tells us that only two polarizations of the photon have physical significance. The other two components of the vector potential  $A_\mu$  can be consistently gauged away. A common gauge choice is the Coulomb or radiation gauge  $A_0 = 0$ ,  $\partial_i A^i = 0$ , which leaves no room for the transformation (3.11). Instead of  $A_0 = 0$ , we may also simply exclude  $A_0$  and  $\pi^0$  from the phase space altogether.

In the above derivation we have silently stepped over an issue concerning the canonical bracket itself. In the present example we have been fortunate enough to have only first-class constraints. In general, however, constraints may arise that do not have weakly vanishing Poisson brackets with the other constraints. These are called *second-class* constraints. If the conservation in time of some constraints places restrictions on the Lagrange multipliers, this is indicative of the presence of second-class constraints. Whereas the freedom in the Lagrange multipliers of the first-class constraints led to an arbitrary contribution to the equations of motion, the multipliers of second-class constraints are not arbitrary, and the latter do not (in general) generate gauge transformations. Now the fundamental point to be made about first-class versus second-class constraints is that (after deriving the full set of constraints) those that are first-class can be set to zero either before or after evaluating the Poisson bracket, as they do not alter the physical content of a state. This is not the case for second-class constraints. The problem is solved, however, by introducing the Dirac bracket

$$\{A, B\}^* = \{A, B\} - \{A, \chi_\alpha\} \{\chi_\alpha, \chi_\beta\}^{-1} \{\chi_\beta, B\}, \quad (3.13)$$

where  $\chi_\alpha$  are the second-class constraints, and they are assumed to be independent. This can always be achieved. The bracket (3.13) enables one to set  $\chi_\alpha = 0$  either before or after its evaluation, since the Dirac bracket of second-class constraints with other functions on phase space vanishes weakly. Moreover, if either  $A$  or  $B$  is first-class, the bracket reduces weakly to the Poisson bracket. The Hamiltonian, in particular, is first-class by construction, and Hamiltonian evolution is unaffected by the replacement of Poisson brackets by Dirac brackets.

## Chapter 4

# Hamiltonian Formulation of General Relativity

In classical mechanics, the Hamiltonian generates time translations of the canonical variables through the Poisson bracket. Similarly, the diffeomorphisms of general relativity are generated by functions that differ from the Hamiltonian only through a simple substitution of vectors.

The asymptotically anti-de Sitter space-times have a boundary at spatial infinity, which gives a contribution to the Hamiltonian referred to as the *surface charge*. In the present chapter, we derive this surface charge starting from the Lagrangian formulation of general relativity with a cosmological constant. Section 4.1 first introduces the notions of ADM decomposition and extrinsic curvature. The former is useful for defining the Hamiltonian; the latter for deriving the surface term in the action, which is done in section 4.2. Section 4.3 finally brings us to the surface charge, which is shown to become the generator of diffeomorphisms, as the bulk term of the Hamiltonian vanishes weakly. Another form of the surface charge, coming from [2], is given.

### 4.1 ADM Decomposition

The Hamiltonian formulation of general relativity makes use of a special decomposition of the metric, called the Arnowitt-Deser-Misner decomposition [23] (ADM for short), which brings out the components parallel and orthogonal to hypersurfaces at constant  $t$ . The decomposition is useful for separating the canonical variables into constraints and physically meaningful quantities, analogous to what happens in Maxwell theory.

Suppose we have a global time function  $t(x^\mu)$  and a vector field  $t^\mu$  obeying  $t^\mu \partial_\mu t = 1$ . In particular, we can introduce a coordinate system  $(t, x^a)$  with as time coordinate the global time function  $t$ . The fact that  $t$  is global implies that we can foliate our space-time into hypersurfaces at constant  $t$ , and  $t^\mu$  generates the flow from the initial spacelike hypersurface  $\Sigma_0$  to  $\Sigma_t$ . If we choose  $t$  to be one of the coordinates of a frame, we simply have  $t^\mu = (\frac{\partial}{\partial t})^\mu$ , since in this case  $\frac{\partial x^\mu}{\partial t} \frac{\partial t}{\partial x^\mu} = 1$ . From now on we will treat  $t$  as a coordinate. We decompose the vector  $t^\mu$  in terms of the vector  $n^\mu$  normal to surfaces at constant  $t$  and an

additional vector  $N^\mu$ :

$$t^\mu \equiv Nn^\mu + N^\mu. \quad (4.1)$$

$N$  is the so-called lapse function and  $N^\mu$  the shift vector. The decomposition (4.1) shows that the lapse function is the normal component and the shift vector is the parallel component of  $t^\mu$  to surfaces at constant  $t$ .

We can use lapse and shift to obtain the ADM form of the metric starting from the following metric adapted to the hypersurface,

$$ds^2 = -(dx^\perp)^2 + \omega_{ab}(x^\perp, x^a)dx^a dx^b, \quad (4.2)$$

where  $a$  and  $b$  run from 1 to  $d-1$ , and the coordinates  $x^\perp, x^a$  are independent. This metric is adapted to the family of hypersurfaces at constant  $t$  in the sense that the coordinates are so defined that the component of  $n^\mu$  in the  $x^\perp$ -direction becomes 1, while the components in the  $x^a$ -directions vanish. Since we have  $t^\mu = (\frac{\partial}{\partial t})^\mu$ , the lapse and shift become

$$\begin{aligned} N &= \frac{\partial x^\perp}{\partial t} \\ N^a &= \frac{\partial x^a}{\partial t}. \end{aligned} \quad (4.3)$$

in the hypersurface-adapted frame. We can use this to turn to another frame  $(t(x^\perp, x^a), x^a)$ . The tensor transformation rule (denoting the coordinates appearing in (4.2) collectively by  $\hat{x}^\rho = (x^\perp, x^a)$ , using a Greek index, and writing  $x^\mu = (t, x^a)$ )

$$g_{\mu\nu}(t, x^a) = \frac{\partial \hat{x}^\rho}{\partial x^\mu} \frac{\partial \hat{x}^\sigma}{\partial x^\nu} \hat{g}_{\rho\sigma}(x^\perp, x^a) \quad (4.4)$$

then tells us that

$$g_{tt} = -N^2 + \omega_{ab}N^aN^b, \quad g_{ta} = \omega_{ab}N^b, \quad g_{ab} = \omega_{ab}, \quad (4.5)$$

yielding the following line element in the coordinates  $t, x^a$ :

$$ds^2 = -N^2 dt^2 + \omega_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \quad (4.6)$$

This form of the metric is called the ADM decomposition.

Note that, due to the possible dependence of the metric on  $x^\perp$  and  $t$ , neither are necessarily global coordinates. This will only be the case for specific solutions with Killing vectors  $\frac{\partial}{\partial x^\perp}$  or  $\frac{\partial}{\partial t}$ , respectively. We called  $t$  a *global* time function only because the space-time can be foliated into hypersurfaces at constant  $t$ , which is indicative of a product topology.

For space-times with the appropriate product topology, the decomposition can also be done with respect to other foliations.

In the following, we will also need the notion of an induced metric. The line element induced on hypersurfaces of constant  $t$  is in the present example

$$ds^2 = \omega_{ab}N^aN^b dt^2 + 2\omega_{ab}N^a dt dx^b + \omega_{ab}dx^a dx^b \quad (4.7)$$

This follows from the more general definition of an induced metric,

$$\gamma_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu, \quad (4.8)$$

with  $n^\mu$  the unit normal to a spacelike or timelike hypersurface, respectively. In order for volume elements to keep the right orientation,  $n^\mu$  should be inward pointing if it is timelike and outward pointing if it is spacelike [24]. If part of the boundary is, for instance, a timelike hypersurface at constant radius, then for this part the normal is

$$\begin{aligned} n^\mu &= \frac{1}{\sqrt{g^{rr}}} g^{\mu(r)}, & n_\mu &= \frac{1}{\sqrt{g^{rr}}} \delta_\mu^{(r)}, \\ n_\mu n^\mu &= 1, \end{aligned} \quad (4.9)$$

and the radial contravariant components of the induced metric vanish. We keep  $d$ -valued indices, however, because as we saw the covariant components are generally nonvanishing.

Indices of tensors on the boundary, including the induced metric, are still to be raised and lowered with the full space-time metric  $g_{\mu\nu}$ . However,  $\gamma^{\mu\nu}$  will not be the inverse of  $\gamma_{\mu\nu}$ , which is noninvertible since we have removed the components in the normal direction. Note that we have  $\sqrt{-g} = N\sqrt{\pm\gamma}$ , with  $N$  the appropriate lapse function. Moreover,  $\gamma_{\mu\nu}n^\mu = 0$ .

A final notion we introduce here is that of the extrinsic curvature of a hypersurface. Its definition is given in terms of the induced metric  $\gamma_{\mu\nu}$  and the normal vector  $n^\mu$  as

$$\Theta_{\mu\nu} = \gamma^\rho_\mu \gamma^\sigma_\nu D_\rho n_\sigma, \quad (4.10)$$

where the covariant derivative contains the full  $d$ -dimensional connection. The  $\gamma^\rho_\mu = g^{\rho\nu}\gamma_{\nu\mu} = \delta^\rho_\mu \pm n^\rho n_\mu$  work as projection operators of tensors onto the hypersurface. If  $n^\mu$  is a geodesic normal, the projection in (4.10) is unnecessary. We can write (4.10) in a more intuitive form by making use of the fact that for  $n^\mu$  to be hypersurface orthogonal is equivalent to (square brackets denote antisymmetrization over all indices)

$$n_{[\mu} D_\nu n_{\rho]} = 0, \quad (4.11)$$

a result following from Frobenius's theorem on integral submanifolds. Suppose we are trying to find integral surfaces (i.e. surfaces that are everywhere parallel to the original surface, and span the space) of an  $m$ -dimensional surface with  $d-m$  linearly independent normal covectors  $n_\nu^{(a)}$ ,  $a = 1, \dots, d-m$  living in the cotangent space of a  $d$ -dimensional manifold. The theorem states that in order for this surface to be integrable, such  $n_\nu$  should obey  $D_{[\mu} n_{\nu]} = \sum_b \eta_{[\mu}^{(b)} v_{\nu]}^{(b)}$ , where the  $\eta_\mu^{(b)}$ ,  $b = 1, \dots, d-m$ , lie in the cotangent space orthogonal to the surface, and  $v_\nu$  are some covectors. If, in particular,  $m = d-1$ , we obtain the result (4.11) by substituting  $\eta_\mu = n_\mu$ .

Contracting (4.11) with  $n^\nu$ , we have

$$D_\rho n_\mu + n^\nu n_\mu D_\nu n_\rho = D_\mu n_\rho + n^\nu n_\rho D_\nu n_\mu, \quad (4.12)$$

since  $n^\nu D_\mu n_\nu = \frac{1}{2} D_\mu (n^\nu n_\nu) = 0$ . Furthermore,

$$\begin{aligned} n^\nu n_\mu D_\nu n_\rho &= \mp g^\nu_\mu D_\nu n_\rho \pm \gamma^\nu_\mu D_\nu n_\rho \\ &= \mp D_\mu n_\rho \pm \gamma^\nu_\mu D_\nu n_\rho. \end{aligned} \quad (4.13)$$



If the lower signs apply, substituting this in (4.12) gives

$$2D_\mu n_\rho + 2D_\rho n_\mu = \gamma^\nu_\rho D_\nu n_\mu - \gamma^\nu_\mu D_\nu n_\rho. \quad (4.14)$$

Since one side is symmetric under  $\rho \leftrightarrow \mu$ , while the other side is antisymmetric under this exchange, both sides are equal to zero. The upper signs immediately give

$$\gamma^\nu_\mu D_\nu n_\rho = \gamma^\nu_\rho D_\nu n_\mu, \quad (4.15)$$

and in both cases the expression is symmetric in  $\mu$  and  $\rho$ . At the same time, using  $\gamma^\alpha_\mu = g^\alpha_\mu \pm n^\alpha n_\mu$ , it can be verified that (4.10) can also be written simply as  $\gamma^\alpha_\mu D_\alpha n_\nu$ . Thus, through a relatively complicated procedure we have verified that the extrinsic curvature is actually symmetric in its indices, and we can now write it as the Lie derivative of  $\gamma_{\mu\nu}$  in the normal direction:

$$\Theta_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu} \quad (4.16)$$

This more intuitive form of  $\Theta_{\mu\nu}$  tells us that the extrinsic curvature measures the rate of change of the induced metric when moving off the hypersurface.  $\Theta_{\mu\nu}$  will turn out to be a useful quantity in defining the Einstein-Hilbert action for space-times with boundary, as is discussed in the next section.

## 4.2 Surface Term in the Action

In order to pass to the Hamiltonian formulation, we first need to know which action to use. For a space-time  $M$  without boundary, Einstein's equations can be derived from the action

$$S = \frac{1}{16\pi G} \int_M \sqrt{-g} (R - 2\Lambda) d^d x \quad (4.17)$$

by varying with respect to the metric  $g_{\mu\nu}$ . Here,  $G$  is Newton's constant in  $d$  dimensions,  $R$  the Ricci scalar curvature, and  $\Lambda$  the cosmological constant. We do not include matter terms, since we will be interested in vacuum solutions.

For a space-time with boundary, the action (4.17) also gives Einstein's equation as long as boundary terms are ignored in the variational principle. However, we wish to use a different variational principle, in which we only set  $\delta g_{\mu\nu} = 0$  at the boundary, but do not throw away surface terms in the variation of the action. These can still arise because we do not demand the derivative of  $g_{\mu\nu}$  to vanish at the boundary.

To determine the boundary term in the action, we will write out the general variation of the Einstein-Hilbert action, and see which term is needed to cancel the unwanted surface term. It is useful to write the Riemann curvature tensor as

$$R^\sigma_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\kappa} \Gamma^\kappa_{\mu\rho} - \Gamma^\sigma_{\rho\kappa} \Gamma^\kappa_{\mu\nu}, \quad (4.18)$$

so that we can recognize its variation,

$$\partial_\nu \delta \Gamma^\sigma_{\mu\rho} - \partial_\rho \delta \Gamma^\sigma_{\mu\nu} + \Gamma^\kappa_{\mu\rho} \delta \Gamma^\sigma_{\nu\kappa} + \Gamma^\sigma_{\nu\kappa} \delta \Gamma^\kappa_{\mu\rho} - \Gamma^\sigma_{\rho\kappa} \delta \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\mu\nu} \delta \Gamma^\sigma_{\rho\kappa}, \quad (4.19)$$

as

$$D_\nu \delta \Gamma^\sigma_{\mu\rho} - D_\rho \delta \Gamma^\sigma_{\mu\nu}. \quad (4.20)$$

The variation of (4.17) thus becomes

$$\begin{aligned} \delta S &= \frac{1}{16\pi G} \int_M \delta(R^\sigma_{\mu\nu\rho} \sqrt{-g} \delta^\nu_\sigma g^{\mu\rho}) - 2\Lambda \delta(\sqrt{-g}) d^d x \\ &= \frac{1}{16\pi G} \int_M \sqrt{-g} \{ \delta^\nu_\sigma g^{\mu\rho} (D_\nu \delta \Gamma^\sigma_{\mu\rho} - D_\rho \delta \Gamma^\sigma_{\mu\nu}) \\ &\quad + \delta^\nu_\sigma R^\sigma_{\mu\nu\rho} \delta g^{\mu\rho} - \frac{1}{2} \delta^\nu_\sigma g^{\mu\rho} R^\sigma_{\mu\nu\rho} g_{\alpha\beta} \delta g^{\alpha\beta} + \Lambda g_{\alpha\beta} \delta g^{\alpha\beta} \} d^d x \end{aligned} \quad (4.21)$$

The term on the second line is easily recognized as a total derivative, and applying Gauss's law

$$\int_M \sqrt{-g} D_\alpha v^\alpha d^d x = \int_{\partial M} \sqrt{\pm \gamma} n_\alpha v^\alpha d^{d-1} x \quad (4.22)$$

where  $\gamma$  is the determinant of the metric induced on the boundary, and  $n_\alpha$  a unit normal to this boundary, now yields

$$\begin{aligned} \delta S &= \frac{1}{16\pi G} \int_M \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu} d^d x \\ &\quad + \frac{1}{16\pi G} \int_{\partial M} \sqrt{\pm \gamma} (g^{\mu\rho} \delta \Gamma^\nu_{\mu\rho} - g^{\mu\nu} \delta \Gamma^\rho_{\mu\rho}) n_\nu d^{d-1} x. \end{aligned} \quad (4.23)$$

The first term alone would give Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (4.24)$$

The remaining term is further reduced using

$$\delta \Gamma^\nu_{\mu\rho} = \frac{1}{2} g^{\nu\sigma} (D_\mu \delta g_{\sigma\rho} + D_\rho \delta g_{\mu\sigma} - D_\sigma \delta g_{\mu\rho}), \quad (4.25)$$

which tells us that

$$\begin{aligned} &g^{\mu\rho} \delta \Gamma^\nu_{\mu\rho} - g^{\mu\nu} \delta \Gamma^\rho_{\mu\rho} \\ &= \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) (D_\mu \delta g_{\rho\sigma} + D_\rho \delta g_{\mu\sigma} - D_\sigma \delta g_{\mu\rho}) \\ &= g^{\mu\nu} g^{\rho\sigma} (D_\sigma \delta g_{\mu\rho} - D_\mu \delta g_{\rho\sigma}), \end{aligned} \quad (4.26)$$

since  $D_\rho \delta g_{\mu\sigma}$  is symmetric under  $\mu \leftrightarrow \sigma$ , while  $D_\mu \delta g_{\rho\sigma} - D_\sigma \delta g_{\mu\rho}$  is antisymmetric under this exchange, as is the prefactor. This leads to

$$\begin{aligned} n_\nu (g^{\mu\rho} \delta \Gamma^\nu_{\mu\rho} - g^{\mu\nu} \delta \Gamma^\rho_{\mu\rho}) &= n^\mu g^{\rho\sigma} (D_\sigma \delta g_{\mu\rho} - D_\mu \delta g_{\rho\sigma}) \\ &= n^\mu \gamma^{\rho\sigma} (D_\sigma \delta g_{\mu\rho} - D_\mu \delta g_{\rho\sigma}) \\ &= -n^\mu \gamma^{\rho\sigma} D_\mu \delta g_{\rho\sigma}. \end{aligned} \quad (4.27)$$

To get the third line we again used the antisymmetry of  $D_\sigma \delta g_{\mu\rho} - D_\mu \delta g_{\rho\sigma}$  under  $\mu \leftrightarrow \sigma$ , so that  $n^\mu n^\rho n^\sigma (D_\sigma \delta g_{\mu\rho} - D_\mu \delta g_{\rho\sigma}) = 0$ . The last equality follows from

the fact that  $\gamma^{\rho\sigma} D_\sigma$  is the covariant derivative along the boundary, where  $\delta g_{\mu\rho}$  vanishes everywhere. We have now written the boundary term in a form which allows us to recognize it as the variation of a tensor living on the boundary. Recall that we had

$$\begin{aligned}\Theta_{\mu\nu} &= \gamma^\rho_\mu \gamma^\sigma_\nu D_\rho n_\sigma \\ &= \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}.\end{aligned}\tag{4.28}$$

The trace of the extrinsic curvature of the boundary is then given by

$$\Theta = \gamma^{\rho\sigma} D_\rho n_\sigma = \gamma^{\rho\sigma} \partial_\rho n_\sigma - \gamma^{\rho\sigma} \Gamma^\mu_{\rho\sigma} n_\mu.\tag{4.29}$$

From the boundary condition  $\delta g_{\mu\nu} = 0$  and the fact that a variation of  $\gamma_{\mu\nu}$  cannot cancel a variation of  $n_\mu$ , it follows that  $\delta n_\mu = \delta \gamma_{\mu\nu} = 0$  at  $\partial M$ . Hence we have

$$\delta\Theta = -\gamma^{\rho\sigma} \delta\Gamma^\mu_{\rho\sigma} n_\mu\tag{4.30}$$

Using (4.25) this reduces to

$$\delta\Theta = \frac{1}{2} n^\mu \gamma^{\rho\sigma} D_\mu \delta g_{\rho\sigma},\tag{4.31}$$

or  $-\frac{1}{2}$  times the surface term in the variation of  $\int_M \sqrt{-g} (R - 2\Lambda) d^d x$ . This finally motivates the use of the adapted Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int_M \sqrt{-g} (R - 2\Lambda) d^d x + \frac{1}{8\pi G} \int_{\partial M} \sqrt{\pm\gamma} \Theta d^{d-1}x.\tag{4.32}$$

This action gives the Einstein equations of motion both for space-times without boundary and for space-times with boundary under the condition that  $\delta g_{\mu\nu} = 0$  at  $\partial M$ , and taking into account surface terms. The action (4.32) is the one we will be working with.

### 4.3 Hamiltonian Formalism

The ADM decomposition now comes in handy, because it allows us to write the action (4.32) in terms of the metric  $h_{\mu\nu}$  induced on surfaces of constant  $t$  and only its first time derivative  $\dot{h}_{\mu\nu}$ , since these two quantities contain all the physical information necessary to describe the dynamics. This in turn allows us to define a canonical momentum

$$\pi^{\mu\nu} \equiv \frac{\delta S}{\delta \dot{h}_{\mu\nu}}.\tag{4.33}$$

Passage to the Hamiltonian formulation will then be achieved through the definition

$$S \equiv \int_{t_i}^{t_f} \left( \int_{\Sigma_t} \pi^{\mu\nu} \dot{h}_{\mu\nu} - H d^{d-1}x \right) dt.\tag{4.34}$$

This shows that properly defining the Hamiltonian is only possible if the space-time has a global time coordinate, and can be foliated into surfaces  $\Sigma_t$  at constant  $t$ :

$$M = [t_i, t_f] \times \Sigma_t. \quad (4.35)$$

Assuming our space-time knows of such a thing as spatial infinity (and the spatial part is not, for instance, a torus or the surface of a sphere), the boundary  $\partial M$  consists of initial and final spacelike boundaries, and a timelike boundary at spatial infinity:

$$\partial M = \Sigma_{t_i} + \Sigma_{t_f} + \Sigma^\infty \quad (4.36)$$

Thus, the action (4.32) becomes

$$S = \frac{1}{16\pi G} \int_M \sqrt{-g} (R - 2\Lambda) d^d x + \frac{1}{8\pi G} \int_{\Sigma_\infty} \sqrt{-h} \Theta d^{d-1} x \\ + \frac{1}{8\pi G} \int_{\Sigma_{t_i}} \sqrt{\gamma} K d^{d-1} x - \frac{1}{8\pi G} \int_{\Sigma_{t_f}} \sqrt{\gamma} K d^{d-1} x, \quad (4.37)$$

with  $\gamma_{\mu\nu}$  the metric induced on the  $\Sigma_{t_i}$  and  $\Sigma_{t_f}$  boundaries, and  $K$  the trace of their extrinsic curvature. Using the contracted Gauss-Codacci relation

$$^{(d-1)}\mathcal{R} = R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} - \Theta^2 + \Theta_{\mu\nu} \Theta^{\mu\nu} \quad (4.38)$$

to express the Ricci curvature scalar  $^{(d-1)}\mathcal{R}$  belonging to  $h_{\mu\nu}$  in terms of the full  $d$ -dimensional curvature  $R_{\mu\nu\rho\sigma}$ , the action (4.37) can be written without a contribution from the spacelike boundaries<sup>1</sup>:

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} \left( ^{(d-1)}\mathcal{R} - 2\Lambda - \Theta^2 + \Theta_{\mu\nu} \Theta^{\mu\nu} \right) d^d x \\ + \frac{1}{8\pi G} \int_{\Sigma_\infty} \sqrt{-\gamma} (K - u_\mu n^\nu D_\nu n^\mu) d^{d-1} x. \quad (4.39)$$

Here,  $u^\mu$  is the unit normal vector to the  $\Sigma^\infty$  boundary, and  $D_\nu$  is still the  $d$ -dimensional covariant derivative. The surface  $\Sigma^\infty$  can again be split up into hypersurfaces  $\Sigma_t^\infty$  at constant  $t$  (in three dimensions these are just closed lines). Using

$$\Theta_{\mu\nu} = \frac{1}{2N} (\dot{h}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu), \quad (4.40)$$

we find as the momentum conjugate to  $h_{\mu\nu}$

$$\pi^{\mu\nu} \equiv \frac{\delta S}{\delta \dot{h}_{\mu\nu}} \\ = \frac{1}{16\pi G} \sqrt{h} (\Theta^{\mu\nu} - \Theta h^{\mu\nu}). \quad (4.41)$$

(the purely spatial metric  $h_{\mu\nu}$  will have a positive determinant).

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<sup>1</sup>We assume the  $\Sigma^\infty$  and  $\Sigma_t$  hypersurfaces to be orthogonal at infinity, which may not always be true, but holds for the asymptotically anti-de Sitter space-times which we are interested in.

For a space-time with the appropriate product topology, the Hamiltonian then becomes [25]

$$H(N) = \int_{\Sigma_t} (N\mathcal{H} + N_\mu \mathcal{H}^\mu) d^{d-1}x - \int_{\Sigma_t^\infty} \sqrt{\sigma} \left( \frac{1}{8\pi G} N\theta - \frac{2}{\sqrt{h}} s_\mu \pi^{\mu\nu} N_\nu \right) d^{d-2}x. \quad (4.42)$$

where  $\sigma$  is the determinant of the induced metric on  $\Sigma_t^\infty$ ,  $s^\mu$  the unit normal vector to this surface (i.e., it is the normal to  $\Sigma^\infty$  projected onto  $\Sigma_t$ ), and  $\theta$  the trace of its extrinsic curvature.  $\mathcal{H}$  and  $\mathcal{H}^\mu$  are the constraints of general relativity, and  $N$  and  $N_\mu$  act as their Lagrange multipliers. The constraints take the form

$$\begin{aligned} \mathcal{H} &= {}^{(d-1)}\mathcal{R} - 2\Lambda + \Theta^2 - \Theta_{\mu\nu}\Theta^{\mu\nu} \\ \mathcal{H}^\mu &= \nabla_\nu \Theta^\nu_\mu - \nabla_\mu \Theta, \end{aligned} \quad (4.43)$$

where  $\nabla_\mu$  is the covariant derivative belonging to  $h_{\mu\nu}$ . The dynamical equations now read

$$\begin{aligned} \dot{h}_{\mu\nu} &= \{h_{\mu\nu}, H(N)\}^*, \\ \dot{\pi}^{\mu\nu} &= \{\pi^{\mu\nu}, H(N)\}^*, \end{aligned} \quad (4.44)$$

for which the explicit expressions are given in Table 4.1, together with a summary of the analogy between source-free Maxwell theory and general relativity with a cosmological constant.

	Constraints	Evolution Equations
General Relativity	${}^{(d-1)}\mathcal{R} + \Theta^2 - \Theta^{\mu\nu}\Theta_{\mu\nu} = 2\Lambda$ $\nabla_\nu \Theta^\nu_\mu - \nabla_\mu \Theta = 0$	$\partial_t \gamma_{\mu\nu} = -2N\Theta_{\mu\nu} + \nabla_\mu N_\nu + \nabla_\nu N_\mu$ $(\partial_t - N^\rho \partial_\rho) \Theta_{\mu\nu} = -\nabla_\mu \nabla_\nu N - N\gamma_{\mu\nu} \Lambda + N({}^{(d-1)}\mathcal{R}_{\mu\nu} + \Theta\Theta_{\mu\nu} - 2\Theta_{\mu\rho}\Theta^\rho_\nu) + \Theta_{\rho\nu}\partial_\mu N^\rho + \Theta_{\mu\rho}\partial_\nu N^\rho$
Maxwell Theory	$\vec{\nabla} \cdot \vec{E} = 0$ $\vec{\nabla} \cdot \vec{B} = 0$	$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}$ $\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}$

Table 4.1: The analogy between Maxwell theory and general relativity.

If initially  $\mathcal{H} = \mathcal{H}^\mu = 0$ , and the evolution equations hold everywhere, then the constraints will be satisfied at all times. Using the ADM decomposition, we have in effect made a change of variables from  $(g_{\mu\nu}, \Gamma^\mu_{\nu\rho})$  to  $(h_{\mu\nu}, \pi^{\mu\nu}, \mathcal{H}, \mathcal{H}^\mu, N, N^\mu)$ , separating physically meaningful quantities from constraints.

Since  $N_\mu$  effectively has only  $d - 1$  components, it is seen that the extended variational principle reduces the degrees of freedom by  $d$ . Starting out

with  $\frac{d(d-1)}{2}$  components of the symmetric,  $(d-1)$ -dimensional tensor  $h_{\mu\nu}$ , the graviton then has  $\frac{d(d-3)}{2}$  physical polarizations, or 2 in  $d = 4$ . Also, pure three-dimensional gravity has no dynamical degrees of freedom at all.

Something else that comes to our attention is that the Hamiltonian of general relativity is weakly zero if boundary terms are ignored. This is in fact characteristic of generally covariant systems for which the canonical variables transform as scalars under time reparametrizations. This is easily seen [26] from

$$S = \int (p_\mu \dot{q}^\mu - \lambda^m \phi_m - H_{\text{rest}}(p, q)) dt \quad (4.45)$$

where  $\phi_m$  are the first- and second-class constraints,  $\lambda^m$  their Lagrange multipliers, and  $H_{\text{rest}}(p, q)$  represents any part of the Hamiltonian that does not vanish weakly. The action (4.45) is invariant under the transformation  $t \rightarrow t - \epsilon(t)$  provided the various variables transform as

$$\begin{aligned} \delta q &= \dot{q} \epsilon \\ \delta p &= \dot{p} \epsilon \\ \delta \lambda^m &= \dot{\lambda}^m \epsilon + \lambda^m \dot{\epsilon}, \end{aligned} \quad (4.46)$$

and only if  $H_{\text{rest}}(p, q) = 0$ . Being a function of  $p$  and  $q$ ,  $H_{\text{rest}}(p, q)$  will transform as a scalar, and there is no prefactor to cancel the transformation as there is for the constraints. Therefore, the action will not be invariant unless  $H_{\text{rest}} = 0$ . This means that time evolution in general relativity consists of gauge transformations.

The last term in (4.42) is the surface charge

$$J(N) = \int_{\Delta_t^\infty} \sqrt{\sigma} \left( \frac{1}{8\pi G} N \theta - \frac{2}{\sqrt{h}} s_\mu \pi^{\mu\nu} N_\nu \right) d^{d-2}x. \quad (4.47)$$

As opposed to the procedure described above, which starts out from the Lagrangian formulation of general relativity, Brown and Henneaux [2] derived the surface charge for 2 + 1-dimensional space-time by demanding the Hamiltonian to have well-defined functional derivatives with respect to the canonical variables. This is needed for the Hamiltonian to be a well-defined generator through the Poisson bracket with the variational principle that takes boundary terms into account. Just like in the Lagrangian case, this method, called the Regge-Teitelboim method, can only determine the surface charge up to a ‘constant’, which they chose such that the charge vanishes for anti-de Sitter space-time. It becomes [2]:

$$J(N) = \frac{1}{16\pi G} \lim_{r \rightarrow \infty} \int \{ \bar{G}^{ijkl} [N^\perp \bar{\nabla}_k h_{ij} - \partial_k N^\perp (h_{ij} - \bar{h}_{ij})] + 2N^i \pi^l_i \} dx_l \quad (4.48)$$

with

$$\bar{G}^{ijkl} = \frac{1}{2} \sqrt{\bar{h}} (\bar{h}^{ik} \bar{h}^{jl} + \bar{h}^{il} \bar{h}^{jk} - 2\bar{h}^{ij} \bar{h}^{kl}) \quad (4.49)$$

where Latin indices denote parallel directions in the hypersurface-adapted frame, and the bar is used for quantities evaluated in anti-de Sitter space. This is

equivalent [25] to subtracting the anti-de Sitter background from (4.47) as

$$J(N) = \int_{\Delta_t^\infty} \frac{1}{8\pi G} (N\sqrt{\sigma}\theta - \bar{N}\sqrt{\bar{\sigma}}\bar{\theta}) - \frac{2}{\sqrt{h}} s_\mu \pi^{\mu\nu} N_\nu \, d^{d-2}x, \quad (4.50)$$

where  $\bar{\theta}$  is the trace of the extrinsic curvature of the  $\Delta_t^\infty$  hypersurface in anti-de Sitter space. Notice that we can do this because  $\bar{N}\sqrt{\bar{\sigma}}\bar{\theta}$  is constant in the sense that we define its derivative with respect to any of the canonical variables to vanish.

Even though the surface charges are not the usual Noether charges (see the Appendix) they do generate transformations. We have seen that the Hamiltonian in the form (4.42) generates time translations. If we want to generate transformations in the direction of a general vector

$$\xi^\mu = \xi^\perp n^\mu + \xi^\parallel{}^\mu, \quad (4.51)$$

(which has the components  $\xi^t = \frac{1}{N}\xi^\perp$ ,  $\xi^a = -\frac{N^a}{N}\xi^\perp + \xi^\parallel{}^a$  in the  $(t, x^a)$  coordinate system) all we have to do is make the replacements  $N \rightarrow \xi^\perp$ ,  $N^\mu \rightarrow \xi^\parallel{}^\mu$ . For instance, working on  $h_{\mu\nu}$ , we have

$$\begin{aligned} \{h_{\mu\nu}, H(\xi)\} &= \mathcal{L}_{\xi^\perp n} h_{\mu\nu} + h^\alpha{}_\mu h^\beta{}_\nu \mathcal{L}_{\xi^\parallel} h_{\alpha\beta} \\ &= \mathcal{L}_\xi h_{\mu\nu}. \end{aligned} \quad (4.52)$$

As discussed at the end of Chapter 3, we can set the constraints to zero,

$$\mathcal{H} = 0, \quad \mathcal{H}_\mu = 0, \quad (4.53)$$

and replace Poisson brackets by Dirac brackets, so that the realization of the diffeomorphism algebra reduces to just the surface charges  $J(\xi)$ . The group property is expressed by

$$\{J(\eta), J(\xi)\}^* = J([\eta, \xi]), \quad (4.54)$$

or in case of a central extension,

$$\{J(\eta), J(\xi)\}^* = J([\eta, \xi]) + K(\eta, \xi), \quad (4.55)$$

where  $K(\eta, \xi)$  is the central term. Note that  $J(\xi)$  is not actually a proper generator of diffeomorphisms, since its functional derivative with respect to  $h_{\mu\nu}$  and  $\pi^{\mu\nu}$  is not well-defined. Therefore, we pretend that we have written the full  $H(\xi)$  (including constraints) in (4.55), then taken functional derivatives, and set the constraints to zero afterwards.

The Hamiltonian with surface charge generates diffeomorphisms with either variational principle - taking into account boundary terms or ignoring them. The Hamiltonian without surface charge, on the other hand, can only be used as a generator if we neglect boundary terms. Therefore, the formulation with surface charge is the appropriate one for describing effects on the boundary.

## Chapter 5

# Geometry

In this chapter we give a short overview of the geometrical properties of anti-de Sitter space, which is the maximally symmetric solution of Einstein's equations with negative cosmological constant. We also treat the BTZ black hole [12], which is a  $(2+1)$ -dimensional solution with negative cosmological constant and a point source at the origin. Asymptotically, the BTZ solution reduces to anti-de Sitter space, as might be expected. However, the BTZ black hole metric is actually asymptotically anti-de Sitter in a stronger sense. This will become clear further on.

### 5.1 Anti-de Sitter Space-Time

Anti-de Sitter space in general space-time dimension  $d$  can most easily be defined as a hyperbolic hypersurface embedded in flat  $(d+1)$ -dimensional space with metric  $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1, \dots, 1)$  via

$$\eta_{\mu\nu} X^\mu X^\nu = -l^2 \quad (5.1)$$

( $\mu, \nu \in (1, \dots, d+1)$ ). The radius of the anti-de Sitter space is  $l$ , which is related to the cosmological constant through  $\Lambda = -\frac{(d-1)(d-2)}{2l^2}$ . Flat space is recovered in the limit  $l \rightarrow \infty$ . The definition (5.1) immediately makes clear that the isometries constitute the group  $SO(d-1, 2)$ , which leaves invariant  $\eta_{\mu\nu} X^\mu X^\nu$ . In terms of the coordinates appearing in (5.1), the generators take the form

$$G_{\mu\nu} = X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu}. \quad (5.2)$$

$G_{\mu\nu}$  is antisymmetric in its indices, so that there are  $\frac{1}{2}d(d+1)$  linearly independent generators, precisely the number of Killing vectors of a maximally symmetric solution in  $d$  dimensions.



For three-dimensional anti-de Sitter space-time (AdS<sub>3</sub>) we can define the coordinates

$$\begin{aligned} t &= l \arctan \left( \frac{X^1}{X^2} \right), \\ r &= (X^1)^2 + (X^2)^2 - l^2, \\ \phi &= \arctan \left( \frac{X^3}{X^4} \right), \end{aligned} \quad (5.3)$$

in terms of which the line element reads

$$ds^2 = - \left( \frac{r^2}{l^2} + 1 \right) dt^2 + \left( \frac{r^2}{l^2} + 1 \right)^{-1} dr^2 + r^2 d\phi^2. \quad (5.4)$$

These coordinates have the ranges  $t \in [0, 2\pi l)$ ,  $\phi \in [0, 2\pi)$ , and  $r \in [0, \infty)$ , i.e., both  $t$  and  $\phi$  are compactified. The space-time thus contains closed timelike curves, which is physically unacceptable because particles will be able to influence themselves by ‘passing through the same point in space-time’ more than once. Therefore, we unwrap the time coordinate  $t$  to become an element of  $(-\infty, \infty)$ , obtaining the so-called universal covering space of AdS<sub>3</sub>. This space-time is commonly referred to simply as AdS<sub>3</sub>. The closely related maximally symmetric solution with positive cosmological constant is called de Sitter space, for which the metric is (5.4) with  $l^2$  replaced by  $-l^2$ .

Locally, one can choose the coordinates

$$\begin{aligned} t &= \frac{-X^2}{l^2(X^1 + X^3)} \\ r &= -\ln \left( \frac{l}{X^1 + X^3} \right) \\ \phi &= \frac{lX^4}{X^1 + X^3} \end{aligned} \quad (5.5)$$

such that

$$ds^2 = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2. \quad (5.6)$$

The coordinates appearing in (5.6) are called Poincaré coordinates. Yet another alternative is

$$ds^2 = l^2 dr^2 + e^{2r} \left( d\phi^2 - \frac{dt^2}{l^2} \right), \quad (5.7)$$

which is obtained by making the replacement  $r \rightarrow e^r$ . The last two metrics show that the boundary at  $r = \infty$  is a flat  $(1+1)$ -dimensional cylinder.

## 5.2 The BTZ Black Hole

The BTZ black hole is a solution of Einstein's equations with negative cosmological constant  $\Lambda = -\frac{1}{l^2}$  and vanishing stress tensor everywhere except at the origin, where there is a point source [13]. In ADM decomposition<sup>1</sup>,

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \quad (5.8)$$

where lapse  $N(r)$  and angular shift  $N^\phi(r)$  are given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2}. \quad (5.9)$$

It can be verified that at large  $r$ , the BTZ black hole solution approaches three-dimensional anti-de Sitter space. This was to be expected, since as  $r \rightarrow \infty$ , we are increasingly far removed from the delta function source. However, the BTZ black hole actually obeys more restrictive asymptotic conditions which will be discussed in the next chapter.

Unlike  $\text{AdS}_3$ , the BTZ black hole metric has only two Killing vectors,  $\frac{d}{dt}$  and  $\frac{d}{d\phi}$ . That these are Killing vectors can immediately be seen from the metric (5.8), which depends neither on  $t$  nor on  $\phi$ . The conserved quantities associated with time translation and rotational symmetry are the mass  $M$  and angular momentum  $J$  of the black hole, respectively. The fact that  $\frac{d}{dt}$  is a symmetry of (5.8) means that the BTZ black hole is a stationary solution.

The inner and outer horizons, where spacelike and timelike vectors interchange roles, are located at the roots of the lapse function:

$$r_{\pm} = l \left( \frac{M}{2} \left( 1 \pm \sqrt{1 - \left( \frac{J}{lM} \right)^2} \right) \right)^{\frac{1}{2}}. \quad (5.10)$$

The BTZ black hole is a spinning black hole. Spinning black holes have a region outside the outer horizon in which particles cannot remain at rest, called the ergosphere. The outer limit of the BTZ ergosphere is located at the surface of infinite redshift  $r = l\sqrt{M}$ , where  $g_{tt}$  vanishes. The extreme black hole is obtained when inner and outer horizons coincide, i.e.,

$$|J| = lM. \quad (5.11)$$

We also see that  $M$  is defined such that it vanishes as  $r_{\pm} \rightarrow 0$ , so that the zero point of energy is reached when the black hole disappears. In the opposite extreme, for  $M \neq 0$ ,  $l \rightarrow \infty$ , we are left with only the inside. The BTZ black hole carries no charge; the charged 2+1 black hole is rather different in that its curvature is not constant [27]. Substituting  $J = 0$ ,  $M = -1$  in the BTZ metric gives back  $\text{AdS}_3$ . However,  $\text{AdS}_3$  cannot be obtained from the BTZ black hole vacuum ( $M = 0$ ,  $J = 0$ ) in a continuous way, since the negative mass solutions with  $M$  between 0 and  $-1$  possess a naked singularity, which is physically unacceptable [27]. This means that the BTZ black hole cannot decay physically

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<sup>1</sup>To clean up the notation, factors of  $8G$  are left implicit in  $M$  and  $J$ .

into  $\text{AdS}_3$ .

Since the BTZ black hole is also a solution of the Einstein equations with vanishing stress-energy tensor (except at the origin), it can however only differ from  $\text{AdS}_3$  by some discrete global identifications. This is because in three space-time dimensions, all local information is stored in the Einstein tensor  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ , and the latter coincides for  $\text{AdS}_3$  and the BTZ black hole outside the source.  $R_{\mu\nu\rho\sigma}$  contains no additional information due to the relation

$$R_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\kappa}\epsilon_{\rho\sigma\lambda}G^{\lambda\kappa} \quad (5.12)$$

which holds in three dimensions. Indeed, it can be verified that  $R_{\mu\nu}$  and  $R_{\mu\nu\sigma\rho}$  each have six independent components in  $d = 3$ . In [27], a procedure is described to obtain the BTZ black hole through identifications in the universal covering space of  $\text{AdS}_3$ . The identification group turns out to be a discrete subgroup of  $SO(2, 2)$ , which is very natural, because in this way, the manifold with the identifications inherits a well-defined local structure.

## Chapter 6

# Central Charge of Asymptotically $\text{AdS}_3$ Space-Times

The presence of a classical central charge in the asymptotic symmetry algebra of certain three-dimensional asymptotically anti-de Sitter space-times was first noted by Brown and Henneaux in 1986 [2]. Often, central charges only arise upon quantization, for instance due to normal ordering of lowering and raising operators, as explained in section 2.3. The fact that a central charge arises already classically in this case has to do with the fact that the asymptotic Killing vectors do not all preserve the boundary metric exactly. The asymptotic symmetries that are actually exact symmetries of the  $\text{AdS}_3$  background do not lead to a central extension. It is necessarily the case that the boundary metric is not preserved by all asymptotic symmetries, because the isometry group turns out to be extended to an infinite-dimensional group ‘at the boundary’ - the conformal group in  $1 + 1$  dimensions. We have seen before that the conformal group in higher dimensions has only a finite number of generators, making this a phenomenon specific to a three-dimensional bulk theory.

Asymptotic symmetries may be naively defined as any diffeomorphisms leading to corrections that are term by term of lower order in  $r$  than the original metric. If we start from the metric (5.6), these include all diffeomorphisms that result (locally) in

$$ds^2 = - \left( \frac{r^2}{l^2} + O(r) \right) dt^2 + \left( \frac{l^2}{r^2} + O \left( \frac{1}{r^3} \right) \right) dr^2 + (r^2 + O(r)) d\phi^2. \quad (6.1)$$

Brown and Henneaux translated (6.1) into conditions on the asymptotic metric by recognizing that at least the  $\text{AdS}_3$  isometry group  $SO(2, 2)$  should be among the asymptotic symmetries.  $SO(2, 2)$  transformations are clearly asymptotic symmetries of  $\text{AdS}_3$  in the above sense. Then, if we first perform an asymptotic symmetry transformation on  $\text{AdS}_3$  that is not contained in  $SO(2, 2)$ , obtaining a different metric that takes the form (6.1), this metric should still be asymptotically invariant under  $SO(2, 2)$  if we want the transformations to form a group. In the dynamical description of these transformations, another con-

dition on the asymptotic metric is that its asymptotic symmetries should have finite canonical generators  $J(\xi)$ .

The asymptotic metric used by Brown and Henneaux takes the form:

$$\begin{aligned}
g_{tt} &= -\frac{r^2}{l^2} + O(1) \\
g_{tr} &= O\left(\frac{1}{r^3}\right) \\
g_{t\phi} &= O(1) \\
g_{rr} &= \frac{l^2}{r^2} + O\left(\frac{1}{r^4}\right) \\
g_{r\phi} &= O\left(\frac{1}{r^3}\right) \\
g_{\phi\phi} &= r^2 + O(1)
\end{aligned} \tag{6.2}$$

where  $t \in (-\infty, \infty)$ ,  $r \in [0, \infty)$ , and  $\phi \in [0, 2\pi)$ . From now on, ‘asymptotically anti-de Sitter’ will be understood to mean (6.2).

This chapter is organized as follows. In section 6.1 we begin by deriving the diffeomorphisms preserving (6.2). These will include  $SO(2, 2)$  as a subgroup, as the asymptotic metric was selected to be invariant under this group. Using the  $SO(2, 2)$  subgroup of the asymptotic Killing vectors, we come back to the form of the asymptotic metric in section 6.2. We show that demanding invariance under  $SO(2, 2)$  leads to conditions that are not quite as strict as (6.2). In section 6.3, the additional restrictions are shown to follow from the demand that the surface charges be finite. The main part of our discussion comes in section 6.4, where we calculate the central charge associated with the asymptotic symmetry group of (6.2) in the canonical formalism. Section 6.5 contains an alternative derivation of the central charge from the work of Balasubramanian and Kraus [28].

## 6.1 Asymptotic Killing Vectors

In this section we derive the asymptotic isometries of (6.2). These have a natural  $SO(2, 2)$  subgroup, as the asymptotic metric was obtained under the very condition that they be  $SO(2, 2)$ -invariant.

Without loss of generality, we can write the candidate vectors in a Laurent expansion in  $r$ . Denoting equal or higher powers of  $r$  by  $o$  and equal or lower powers by  $O^1$ :

$$\begin{aligned}
\xi^t &= o(r) + A(t, \phi) + \frac{B(t, \phi)}{r} + \frac{C(t, \phi)}{r^2} + \frac{D(t, \phi)}{r^3} + O\left(\frac{1}{r^4}\right), \\
\xi^r &= o(r^2) + E(t, \phi)r + F(t, \phi) + O\left(\frac{1}{r}\right), \\
\xi^\phi &= o(r) + G(t, \phi) + \frac{H(t, \phi)}{r} + \frac{I(t, \phi)}{r^2} + \frac{J(t, \phi)}{r^3} + O\left(\frac{1}{r^4}\right).
\end{aligned} \tag{6.3}$$

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<sup>1</sup>We do not write out more terms due to some prior knowledge on what the Killing vectors will turn out to look like from [2] and [9].

Applying Lie transport with these vectors  $\xi^\mu$  to  $g_{\mu\nu}$  and demanding that (6.2) is preserved, results in

$$\begin{aligned}
B = D = F = H = J &= 0, \\
2C &= -l^4 \partial_t E, \\
\partial_\phi A &= l^2 \partial_t G, \\
2I &= l^2 \partial_\phi E, \\
E &= -\partial_\phi G = -\partial_t A.
\end{aligned} \tag{6.4}$$

and all higher order terms should vanish.

The fact that  $[l^2 \partial_t^2 - \partial_\phi^2] A(t, \phi) = 0$  allows us to write  $A(t, \phi)$  as  $l(\eta^+ + \eta^-)$ , where  $\partial_\pm \eta^\mp = 0$  and  $\partial_\pm \equiv \frac{1}{2}(l \partial_t \pm \partial_\phi)$ . The other components then follow uniquely from the relations (6.4), and it can be verified that the asymptotic Killing vectors become

$$\begin{aligned}
\xi^t &= l(\eta^+ + \eta^-) + \frac{l^3}{2r^2}(\partial_+^2 \eta^+ + \partial_-^2 \eta^-) + O\left(\frac{1}{r^4}\right), \\
\xi^r &= -r(\partial_+ \eta^+ + \partial_- \eta^-) + O\left(\frac{1}{r}\right), \\
\xi^\phi &= \eta^+ - \eta^- - \frac{l^2}{2r^2}(\partial_+^2 \eta^+ - \partial_-^2 \eta^-) + O\left(\frac{1}{r^4}\right).
\end{aligned} \tag{6.5}$$

We can see that the  $\eta^+$  and  $\eta^-$  transformations do not talk to each other,

$$\begin{aligned}
[\xi(\eta_1^-), \xi(\eta_2^-)] &= \xi(\eta_1^- \partial_- \eta_2^- - \eta_2^- \partial_- \eta_1^-) \\
[\xi(\eta_1^+), \xi(\eta_2^+)] &= \xi(\eta_1^+ \partial_+ \eta_2^+ - \eta_2^+ \partial_+ \eta_1^+).
\end{aligned} \tag{6.6}$$

Moreover, it is clear that we are dealing with an infinite-dimensional group. Following Strominger [9] we can then let  $l_n$  and  $\bar{l}_n$  denote the generators of the diffeomorphisms with  $\eta^+ = e^{in(\frac{t}{l} + \phi)}$ ,  $\eta^- = 0$  and  $\eta^- = e^{in(\frac{t}{l} - \phi)}$ ,  $\eta^+ = 0$ , respectively. The resulting Lie bracket algebra is the loop algebra

$$\begin{aligned}
[l_m, l_n] &= 2i(m - n)l_{m+n} \\
[\bar{l}_m, \bar{l}_n] &= 2i(m - n)\bar{l}_{m+n} \\
[l_m, \bar{l}_n] &= 0,
\end{aligned} \tag{6.7}$$

and we have indeed found the infinite-dimensional conformal group on the  $(1+1)$ -dimensional boundary.

The  $so(2,2)$  subalgebra is spanned by  $l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1$ . These take the form

$$\begin{aligned} l_{-1}^t &= l e^{-i(\frac{t}{l}+\phi)} - \frac{l^3}{2r^2} e^{-i(\frac{t}{l}+\phi)} \\ l_{-1}^r &= ir e^{-i(\frac{t}{l}+\phi)} \\ l_{-1}^\phi &= e^{-i(\frac{t}{l}+\phi)} + \frac{l^2}{2r^2} e^{-i(\frac{t}{l}+\phi)} \end{aligned} \tag{6.8}$$

$$\begin{aligned} l_0^t &= l \\ l_0^r &= 0 \\ l_0^\phi &= 1 \end{aligned} \tag{6.9}$$

$$\begin{aligned} l_1^t &= l e^{i(\frac{t}{l}+\phi)} - \frac{l^3}{2r^2} e^{i(\frac{t}{l}+\phi)} \\ l_1^r &= -ir e^{i(\frac{t}{l}+\phi)} \\ l_1^\phi &= e^{i(\frac{t}{l}+\phi)} + \frac{l^2}{2r^2} e^{i(\frac{t}{l}+\phi)} \end{aligned} \tag{6.10}$$

$$\begin{aligned} \bar{l}_{-1}^t &= l e^{-i(\frac{t}{l}-\phi)} - \frac{l^3}{2r^2} e^{-i(\frac{t}{l}-\phi)} \\ \bar{l}_{-1}^r &= ir e^{-i(\frac{t}{l}-\phi)} \\ \bar{l}_{-1}^\phi &= -e^{-i(\frac{t}{l}-\phi)} - \frac{l^2}{2r^2} e^{-i(\frac{t}{l}-\phi)} \end{aligned} \tag{6.11}$$

$$\begin{aligned} \bar{l}_0^t &= l \\ \bar{l}_0^r &= 0 \\ \bar{l}_0^\phi &= -1 \end{aligned} \tag{6.12}$$

$$\begin{aligned} \bar{l}_1^t &= l e^{i(\frac{t}{l}-\phi)} - \frac{l^3}{2r^2} e^{i(\frac{t}{l}-\phi)} \\ \bar{l}_1^r &= -ir e^{i(\frac{t}{l}-\phi)} \\ \bar{l}_1^\phi &= -e^{i(\frac{t}{l}-\phi)} - \frac{l^2}{2r^2} e^{i(\frac{t}{l}-\phi)} \end{aligned} \tag{6.13}$$

## 6.2 The Asymptotic Metric

We can now reverse the process, and use the  $SO(2,2)$  vectors given above to derive asymptotic conditions for the metric. These conditions will not yet be quite as restrictive as (6.2). We thus find an asymptotic metric that is consistent under  $SO(2,2)$ , and allows for an even larger asymptotic symmetry group than (6.2).

To derive the form the asymptotic metric should take, we start by looking at the transformations generated by  $l_0$  and  $\bar{l}_0$ :

$$\begin{aligned}\delta_{l_0} g_{\mu\nu} &= (l\partial_t + \partial_\phi)g_{\mu\nu} \\ \delta_{\bar{l}_0} g_{\mu\nu} &= (l\partial_t - \partial_\phi)g_{\mu\nu}.\end{aligned}\tag{6.14}$$

As these variations are of the same order of  $r$  as the metric itself, this tells us that any exact leading order terms in  $r$  should not depend on  $t$  or  $\phi$ . Lie transport of  $g_{tt}$  with  $l_{\pm 1}$  gives

$$\begin{aligned}\delta_{l_{\pm 1}} g_{tt} &= 2g_{tt}\partial_t l_{\pm 1}^t + 2g_{rt}\partial_t l_{\pm 1}^r + 2g_{t\phi}\partial_t l_{\pm 1}^\phi + l_{\pm 1}^t\partial_t g_{tt} + l_{\pm 1}^r\partial_r g_{tt} + l_{\pm 1}^\phi\partial_\phi g_{tt} \\ &= [2(\pm i \mp \frac{il^2}{2r^2})g_{tt} + 2\frac{r}{l}g_{tr} + 2(\pm \frac{i}{l} \pm \frac{il}{2r^2})g_{t\phi} + (l - \frac{l^3}{2r^2})\partial_t g_{tt} \\ &\quad \mp ir\partial_r g_{tt} + (1 + \frac{l^2}{2r^2})\partial_\phi g_{tt}] e^{\pm i(\frac{t}{l} + \phi)}.\end{aligned}\tag{6.15}$$

We can add up the transformations under  $l_1$  and  $l_{-1}$  for  $\frac{t}{l} + \phi = 2\pi n, n \in \mathbb{Z}$ :

$$\delta_{(l_1 + l_{-1})} g_{tt} = 4\frac{r}{l}g_{tr} + (2l - \frac{l^3}{r^2})\partial_t g_{tt} + (2 + \frac{l^2}{r^2})\partial_\phi g_{tt}.\tag{6.16}$$

This tells us that if the leading order of  $g_{tt}$  is to be exactly preserved (so that, as we have just seen, it does not depend on  $t$  or  $\phi$ )  $g_{tr}$  should be of at least two orders lower than  $g_{tt}$ . Subtracting one from the other, we get

$$\delta_{(l_1 - l_{-1})} g_{tt} = (4i - \frac{2l^2}{r^2})g_{tt} + (\frac{4i}{l} + \frac{2il}{r^2})g_{t\phi} - 2ir\partial_r g_{tt}\tag{6.17}$$

showing that, if  $g_{t\phi}$  is lower order,  $g_{tt}$  obeys

$$r\partial_r g_{tt} = 2g_{tt}\tag{6.18}$$

to leading order in  $r$ . Similarly (still with  $\frac{t}{l} + \phi = 2\pi n$ ),

$$\delta_{(l_1 - l_{-1})} g_{t\phi} = (2il - \frac{il^3}{r^2})g_{tt} + 4ig_{t\phi} + (\frac{2i}{l} + \frac{il}{r^2})g_{\phi\phi} - 2ir\partial_r g_{t\phi}\tag{6.19}$$

shows that to leading order

$$g_{\phi\phi} = -l^2 g_{tt}\tag{6.20}$$

if  $g_{t\phi}$  is of lower order than  $g_{tt}$  and  $g_{\phi\phi}$ , and the latter are of the same order. In fact, suppose that either  $g_{tt}$  or  $g_{\phi\phi}$  were of highest order, then that component of the metric should vanish completely not to give high order corrections to  $g_{t\phi}$ .



Therefore, as long as  $g_{t\phi}$  is of lower order than both,  $g_{tt}$  and  $g_{\phi\phi}$  should indeed have the same order, and (6.20) should hold. The transformation

$$\delta_{(l_1-l_{-1})}g_{rr} = -4ig_{rr} - 2ir\partial_r g_{rr} \quad (6.21)$$

gives

$$r\partial_r g_{rr} = -2g_{rr} \quad (6.22)$$

for the exact leading order part of  $g_{rr}$ . As a final example, we look at the transformation of  $g_{\phi\phi}$  under Lie transport with  $l_1 - l_{-1}$ :

$$\delta_{(l_1-l_{-1})}g_{\phi\phi} = (4il - \frac{2il^3}{r^2})g_{t\phi} + (4i + \frac{2il^2}{r^2})g_{\phi\phi} - 2ir\partial_r g_{\phi\phi}. \quad (6.23)$$

From this we derive

$$r\partial_r g_{\phi\phi} = 2g_{\phi\phi} \quad (6.24)$$

for any exact leading order of  $g_{\phi\phi}$ , as long as  $g_{t\phi}$  is lower order. Proceeding in the same way for different combinations of  $l_{\pm 1}$  and  $\bar{l}_{\pm 1}$  and  $\frac{t}{l} = (m+n)\pi$ ,  $\phi = (n-m)\pi$ ,  $n, m \in Z$ , so that all exponentials become equal to 1, we can derive many more relations, each of which is obeyed by the asymptotic metric

$$\begin{aligned} g_{tt} &= -\frac{r^2}{l^2} + O(1) \\ g_{tr} &= O\left(\frac{1}{r}\right) \\ g_{t\phi} &= O(1) \\ g_{rr} &= \frac{l^2}{r^2} + O\left(\frac{1}{r^3}\right) \\ g_{r\phi} &= O\left(\frac{1}{r}\right) \\ g_{\phi\phi} &= r^2 + O(1). \end{aligned} \quad (6.25)$$

These fall-off conditions are consistent under  $SO(2, 2)$ , and the vectors preserving (6.25) obey

$$\begin{aligned} B = F = H &= 0 \\ \partial_\phi A &= l^2 \partial_t G, \\ E = -\partial_\phi G &= -\partial_t A. \end{aligned} \quad (6.26)$$

There are no further restrictions on  $C, D, I$ , and  $J$ . That the anti-de Sitter group is part of the asymptotic symmetries is reflected by the fact that all of these relations also hold for the vectors (6.5). The vectors obeying (6.26) look like

$$\begin{aligned} \xi^t &= l(\eta^+ + \eta^-) + O\left(\frac{1}{r^2}\right), \\ \xi^r &= -r(\partial_+ \eta^+ + \partial_- \eta^-) + O\left(\frac{1}{r}\right), \\ \xi^\phi &= \eta^+ - \eta^- + O\left(\frac{1}{r^2}\right). \end{aligned} \quad (6.27)$$

Thus, it seems that we have found an even larger asymptotic symmetry group than the conformal group. However, we will see in the next section that if we introduce Hamiltonian dynamics, the boundary conditions (6.25) need to be sharpened.

We have seen that the leading order terms of a metric that has  $SO(2, 2)$  in its asymptotic symmetry group are those of anti-de Sitter space-time. This is by no means surprising. However, note that, vice versa, the fact that  $SO(2, 2)$  is part of the asymptotic symmetries does *not* follow from the leading orders of the metric being those of anti-de Sitter space. We can either weaken or strengthen the boundary conditions in such a way that they are no longer preserved by all  $SO(2, 2)$  vectors. As an example, consider

$$\begin{aligned}
g_{tt} &= -\frac{r^2}{l^2} + O(r) \\
g_{tr} &= O(r) \\
g_{t\phi} &= O(r) \\
g_{rr} &= \frac{l^2}{r^2} + O\left(\frac{1}{r^3}\right) \\
g_{r\phi} &= O(r) \\
g_{\phi\phi} &= r^2 + O(r).
\end{aligned} \tag{6.28}$$

It can be verified that the vectors (6.3) should have  $\partial_t E = \partial_\phi E = 0$  for these asymptotic conditions to be consistent. This is the case for  $l_0$  and  $\bar{l}_0$  (which are symmetries whenever the leading order components of the metric do not depend on  $t$  or  $\phi$ ), but not for  $SO(2, 2)$  vectors in general.

### 6.3 Finiteness of the Surface Charges

So far, we have treated the asymptotic symmetries outside a dynamical context. If we introduce Hamiltonian mechanics in the general relativistic setting, the asymptotic symmetries are generated by the surface charges  $J(\xi)$ . We show that the demand that these generators be well-defined leads to additional conditions on the asymptotic metric.

For the asymptotics (6.25), the surface charge (4.48) given in Chapter 4 simplifies to

$$\begin{aligned}
J(\xi) = \frac{1}{16\pi G} \lim_{r \rightarrow \infty} \int \left\{ \frac{l}{r} \xi^\perp + \frac{r^3}{l^3} \xi^\perp (g_{rr} - \frac{l^2}{r^2}) + \frac{1}{l} (\frac{1}{r} \xi^\perp + \partial_r \xi^\perp) (g_{\phi\phi} - r^2) \right. \\
\left. + \frac{1}{l} (\xi^\perp \partial_\phi g_{r\phi} - g_{r\phi} \partial_\phi \xi^\perp) + 2\xi^{\parallel\phi} \pi_\phi^r \right\} d\phi \tag{6.29}
\end{aligned}$$

The term  $\frac{1}{l} (\xi^\perp \partial_\phi g_{r\phi} - g_{r\phi} \partial_\phi \xi^\perp)$  goes to zero for (6.2), because for that metric it is of  $O(\frac{1}{r^2})$ . Here, it is of  $O(1)$ , which would not pose a difficulty if it were not for the fact that  $g_{r\phi}$  obtained via Lie transport of the anti-de Sitter metric with an allowed vector  $\eta^\mu$  (hence yielding an allowed metric),

$$\begin{aligned}
g_{r\phi} &= \bar{g}_{r\phi} + \mathcal{L}_\eta \bar{g}_{r\phi} \\
&= \bar{g}_{rr} \partial_\phi \eta^r + \bar{g}_{\phi\phi} \partial_r \eta^\phi,
\end{aligned} \tag{6.30}$$

contains unspecified orders of  $O(\frac{1}{r})$  due to the arbitrary  $O(\frac{1}{r^2})$  term in  $\eta^\phi$ . The same problem arises with  $\frac{1}{l}(\frac{1}{r}\xi^\perp + \partial_r \xi^\perp)(g_{\phi\phi} - r^2)$ , for

$$\begin{aligned} g_{\phi\phi} &= \bar{g}_{\phi\phi} + \mathcal{L}_\eta \bar{g}_{\phi\phi} \\ &= r^2 + 2\bar{g}_{\phi\phi} \partial_\phi \eta^\phi + \eta^r \partial_r \bar{g}_{\phi\phi} \end{aligned} \quad (6.31)$$

has an undetermined  $O(1)$  term, while  $\frac{1}{r}\xi^\perp + \partial_r \xi^\perp$  is of the same order, so that the product does not vanish in the limit  $r \rightarrow \infty$ . Similarly, the term  $2\xi^\phi \pi_\phi^r$  contains unspecified orders of  $O(1)$  coming from  $O(\frac{1}{r^2})$  in  $\eta^t$ , since  $\pi_\phi^r$  is equal to leading order in  $r$  to  $g_{t\phi}$  [2]. It is clear that we need more restrictions on the asymptotic Killing vectors in order to have well-defined surface charges. These restrictions will come from stricter fall-off conditions for the metric. Specifically,  $2C = -l^4 \partial_t E$  comes from  $g_{tr}$  being of order  $O(\frac{1}{r^2})$  rather than  $O(\frac{1}{r})$ , and  $2I = l^2 \partial_\phi E$  comes from  $g_{r\phi}$  being of order  $O(\frac{1}{r^2})$ . Moreover, the term  $\frac{r^3}{l^3} \xi^\perp (g_{rr} - \frac{l^2}{r^2})$  diverges unless  $g_{rr} = \frac{l^2}{r^2} + O(\frac{1}{r^4})$ , after which consistency even requires  $g_{tr} = O(\frac{1}{r^3})$  and  $g_{r\phi} = O(\frac{1}{r^3})$ . This has as a consequence that  $D = J = 0$  in (6.3) and finally leaves us with (6.2).

## 6.4 Boundary Dynamics and the Central Charge

We mentioned that the Poisson bracket algebra of the asymptotic symmetries of (6.2) turns out to be centrally extended. The resulting algebra looks like

$$\begin{aligned} \{J(l_m), J(l_n)\} &= 2i(m-n)J(l_{m+n}) + \frac{ic}{3}(m^3 - m)\delta_{m+n,0}, \\ \{J(\bar{l}_m), J(\bar{l}_n)\} &= 2i(m-n)J(\bar{l}_{m+n}) + \frac{i\bar{c}}{3}(m^3 - m)\delta_{m+n,0}, \\ \{J(l_m), J(\bar{l}_n)\} &= 0, \end{aligned} \quad (6.32)$$

corresponding to two copies of the Virasoro algebra with an as yet unknown central charge  $c$ . In the present section we calculate this central charge by making use of the canonical formalism introduced in Chapter 4. We will also see that  $c = \bar{c}$ .

As discussed in the previous section, the surface charge  $J(\xi)$  with anti-de Sitter background subtracted takes the form

$$J(\xi) = \frac{1}{16\pi G} \lim_{r \rightarrow \infty} \int d\phi \left\{ \frac{l}{r} \xi^\perp + \frac{r^3}{l^3} \xi^\perp (g_{rr} - \frac{l^2}{r^2}) + \frac{1}{l} (\frac{1}{r} \xi^\perp + \partial_r \xi^\perp) (g_{\phi\phi} - r^2) + 2\xi^\phi \pi_\phi^r \right\}, \quad (6.33)$$

for the given metric (6.2). Since the integration is performed over the  $r \rightarrow \infty$  boundary,  $J(\xi)$  vanishes for any asymptotic Killing vector that is of  $O(\frac{1}{r^4})$  in its  $t$  and  $\phi$  components, and of  $O(\frac{1}{r})$  in its  $r$  component, so that the actual group of asymptotic symmetries may be defined as the factor group obtained by identifying vectors that differ only in these orders in  $\frac{1}{r}$  [2].

As discussed at the end of Chapter 4, the central term is given by the difference between the Dirac bracket algebra of the surface charges and the charge evaluated on the Lie bracket of two asymptotic symmetries:

$$\{J(\xi), J(\eta)\}^* = J([\xi, \eta]) + K(\xi, \eta). \quad (6.34)$$

$K(\xi, \eta)$  is the central term. Now, the value of the charge on the surface deformed by  $\eta^\mu$  is:

$$\begin{aligned} J(\xi) + \delta_\eta J(\xi) &= J(\xi) + \{J(\xi), J(\eta)\}^* \\ &= J(\xi) + J([\xi, \eta]) + K(\xi, \eta). \end{aligned} \quad (6.35)$$

If we start out with the AdS<sub>3</sub> metric  $\bar{g}_{\mu\nu}$ , then both  $J(\xi) = 0$  and  $J([\xi, \eta]) = 0$  (substituting  $g_{\mu\nu} = \bar{g}_{\mu\nu}$  in the expression (6.33) yields zero for any vector), and we are left with<sup>2</sup>

$$K(\xi, \eta) = \delta_\eta J(\xi). \quad (6.36)$$

This can be evaluated by substituting

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \mathcal{L}_\eta \bar{g}_{\mu\nu} \quad (6.37)$$

in the expression for  $J(\xi)$ . Since the surface charge vanishes for anti-de Sitter space, we see that  $K(\xi, \eta)$  would diverge if we had not demanded the surface charges to be finite for all asymptotic Killing vectors. The  $J(\xi)$  would then no longer have formed a projective representation of the asymptotic symmetries.

If we choose  $\xi = l_m$  and  $\eta = l_n$  in (6.36), we have, to leading order in  $r$ :

$$\xi^\perp = N l_m^t \simeq \frac{r}{l} \left( l - \frac{m^2 l^3}{2r^2} \right) e^{im(\frac{t}{l} + \phi)}, \quad (6.38)$$

$$\xi^{\parallel\phi} = l_m^\phi + N^\phi l_m^t \simeq \left( 1 + \frac{m^2 l^2}{2r^2} \right) e^{im(\frac{t}{l} + \phi)}, \quad (6.39)$$

$$\begin{aligned} g_{rr} &= \left( \frac{r^2}{l^2} + 1 \right)^{-1} + 2\bar{g}_{rr} \partial_r l_n^r + L_n^r \partial_r \bar{g}_{rr} \\ &= \left( \frac{r^2}{l^2} + 1 \right)^{-1} + 2 \left( \frac{r^2}{l^2} + 1 \right)^{-1} \partial_r (-inr e^{in(\frac{t}{l} + \phi)}) + -inr e^{in(\frac{t}{l} + \phi)} \partial_r \left( \frac{r^2}{l^2} + 1 \right)^{-1} \\ &\simeq \frac{l^2}{r^2} - \frac{l^4}{r^4} - 2in \frac{l^4}{r^4} e^{in(\frac{t}{l} + \phi)}, \end{aligned} \quad (6.40)$$

$$\begin{aligned} g_{\phi\phi} &= r^2 + 2\bar{g}_{\phi\phi} \partial_\phi L_n^\phi + l_n^r \partial_r \bar{g}_{\phi\phi} \\ &= r^2 + 2r^2 \partial_\phi \left( 1 + \frac{n^2 l^2}{2r^2} \right) e^{in(\frac{t}{l} + \phi)} - 2inr^2 e^{in(\frac{t}{l} + \phi)} \\ &= r^2 + il^2 n^3 e^{in(\frac{t}{l} + \phi)}, \end{aligned} \quad (6.41)$$

and, also to the relevant orders in  $r$ ,  $\pi_\phi^r = g_{t\phi}^r$  [2], with

$$\begin{aligned} g_{t\phi} &= \bar{g}_{tt} \partial_\phi l_n^t + \bar{g}_{\phi\phi} \partial_t l_n^\phi \\ &= - \left( \frac{r^2}{l^2} + 1 \right) \partial_\phi \left( l - \frac{n^2 l^3}{2r^2} \right) e^{in(\frac{t}{l} + \phi)} + r^2 \partial_t \left( 1 + \frac{n^2 l^2}{2r^2} \right) e^{in(\frac{t}{l} + \phi)} \\ &= il(n^3 - n + \frac{n^3 l^2}{2r^2}) e^{in(\frac{t}{l} + \phi)}. \end{aligned} \quad (6.42)$$

---

<sup>2</sup>The Dirac bracket does not vanish, because  $g_{\mu\nu}$  is treated as a variable inside the bracket.

For  $\bar{l}_m, \bar{l}_n$  we obtain similar results. It can be verified that  $J(\xi)$  integrates to zero unless  $m = -n$ , and that, accordingly, the central term is<sup>3</sup>

$$K(l_m, l_n) = K(\bar{l}_m, \bar{l}_n) = \frac{il}{2G}(m^3 - m)\delta_{m+n,0}. \quad (6.43)$$

The central charge appearing in (6.32) thus becomes

$$c = \bar{c} = \frac{3l}{2G}. \quad (6.44)$$

We have found a classical central charge in the Poisson bracket algebra of the asymptotic symmetry group of three-dimensional asymptotically anti-de Sitter space-times. Since the central charge only shows up after introducing Poisson brackets, the central element does not generate any transformation.

The above derivation has immediately shown that the asymptotic symmetries for which  $\mathcal{L}_\eta \bar{g}_{\mu\nu} = 0$  have no associated central term. Hence, the  $SO(2, 2)$  subalgebra of diffeomorphisms generating isometries of anti-de Sitter space is not centrally extended.

Since we obtained two copies of the Virasoro algebra, it seems possible that the boundary dynamics we have described can be captured in a  $(1 + 1)$ -dimensional conformal field theory. Indeed, it was shown in [3] that the boundary degrees of freedom correspond (up to zero modes) to those of Liouville theory. Analogously, the boundary fluctuations in  $\text{AdS}_3$  supergravity have a dual interpretation within super-Liouville theory [4, 5, 6]. We summarize the derivation of [3] in chapter 7.

## 6.5 Brown-York Stress Tensor

Since the original calculation by Brown and Henneaux, there have been many alternative derivations of the central charge of asymptotically  $\text{AdS}_3$ . We discuss one approach by Balasubramanian and Kraus [28], who looked at the transformation properties of the Brown-York stress-energy tensor [29]:

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}}. \quad (6.45)$$

---

<sup>3</sup>Brown and Henneaux used the aforementioned relation  $[l^2 \partial_t^2 - \partial_\phi^2]A(t, \phi) = 0$  (cf. (6.4)) to Fourier decompose the asymptotic Killing vectors, yielding four sets of generators  $A_n, B_n, C_n, D_n$ , with  $A(t, \phi)$  equal to  $l \cos \frac{nt}{l} \cos n\phi$ ,  $l \sin \frac{nt}{l} \sin n\phi$ ,  $l \sin \frac{nt}{l} \cos n\phi$ , and  $l \cos \frac{nt}{l} \sin n\phi$ , respectively. The change of basis from  $A_n, B_n, C_n$  and  $D_n$  to the  $l_n, \bar{l}_n$  used by Strominger, is as follows:

$$\begin{aligned} l_n &= A_n - B_n + iC_n + iD_n, \\ \bar{l}_n &= A_n + B_n + iC_n - iD_n \end{aligned}$$

([2] contains a sign error for  $B_n^r$ ). In Brown-Henneaux's notation, the only nonvanishing central terms are

$$\begin{aligned} K(A_n, C_m) &= 2\pi l m(m^2 - 1)\delta_{|n|, |m|}, \\ K(B_n, D_m) &= -2\pi l m(m^2 - 1)\delta_{|n|, |m|}. \end{aligned}$$

The tensor is called ‘quasilocal’ because it involves the induced boundary metric  $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$  rather than  $g_{\mu\nu}$  itself. Since the Brown-York tensor normally diverges as the boundary is taken to infinity, there have been several attempts to give an unambiguous renormalization procedure. The most prominent approach, which was also suggested by Brown and York, has been to embed the boundary in a reference space-time, and to subtract the boundary term evaluated in this space-time. This is similar to the procedure used above to make the surface charges  $J(\xi)$  vanish for anti-de Sitter space. A problem is, however, that it is not always possible to embed the boundary in some reference space-time. Therefore, Balasubramanian and Kraus have suggested an alternative procedure, adding counterterms to the action that depend only on the boundary metric to render the energy density finite. The procedure is thus independent of any reference space-time.

Let  $\gamma_{\mu\nu}$  denote the induced metric of surfaces at constant  $r$ . Then, if we add a boundary term and counterterm action to the Einstein-Hilbert action as

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} (R - 2\Lambda) d^d x - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{-\gamma} \Theta d^{d-1}x + \frac{1}{8\pi G} S_{ct}(\gamma_{\mu\nu}), \quad (6.46)$$

the stress tensor becomes

$$T_{\mu\nu} = \frac{1}{8\pi G} (\Theta_{\mu\nu} - \Theta \gamma_{\mu\nu} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{\mu\nu}}) \quad (6.47)$$

with  $\Theta_{\mu\nu}$  the extrinsic curvature

$$\Theta_{\mu\nu} = \gamma_{\mu\alpha} \gamma_{\nu\beta} D^\alpha n^\beta \quad (6.48)$$

and  $\Theta$  its trace. Now it turns out on the basis of covariance [28] that for the case of  $\text{AdS}_3$ , the counterterm should be<sup>4</sup>

$$S_{ct} = -\frac{1}{l} \int \sqrt{-\gamma} d^{d-1}x, \quad (6.49)$$

so that

$$T_{\mu\nu} = \frac{1}{8\pi G} (\Theta_{\mu\nu} - \Theta \gamma_{\mu\nu} - \frac{1}{l} \gamma^{\mu\nu}). \quad (6.50)$$

In particular, this causes the stress tensor to vanish for the Poincaré patch of three-dimensional anti-de Sitter space. Therefore we can look at a deformation of this space in order to recognize the central charge. Writing the Poincaré patch of  $\text{AdS}_3$  in terms of  $r$  and the coordinates  $\tau^+ = t + \phi$ ,  $\tau^- = t - \phi$ , we have

$$ds^2 = \frac{l^2}{r^2} dr^2 - r^2 d\tau^+ d\tau^-. \quad (6.51)$$

The asymptotic isometries read

$$\begin{aligned} r &\rightarrow r(1 - \partial_+ \eta^+ - \partial_- \eta^-) \\ \tau^+ &\rightarrow \tau^+ + 2\eta^+ + \frac{l^2}{r^2} \partial_-^2 \eta^- \\ \tau^- &\rightarrow \tau^- + 2\eta^- + \frac{l^2}{r^2} \partial_+^2 \eta^+, \end{aligned} \quad (6.52)$$

---

<sup>4</sup>Additional counterterms are possible, but these will not contribute to the energy-momentum tensor [28].

and they turn the metric into

$$ds^2 = \frac{l^2}{r^2} dr^2 - r^2 d\tau^+ d\tau^- - l^2 (\partial_+^3 \eta^+) (d\tau^+)^2 - l^2 (\partial_-^3) (d\tau^-)^2. \quad (6.53)$$

Now we can calculate

$$\begin{aligned} T_{++} &= -\frac{1}{8\pi G l} \gamma_{++} = \frac{l}{8\pi G} \partial_+^3 \eta^+ \\ T_{--} &= -\frac{1}{8\pi G l} \gamma_{--} = \frac{l}{8\pi G} \partial_-^3 \eta^-. \end{aligned} \quad (6.54)$$

The transformation rule for the stress tensor under the diffeomorphisms  $\tau^+ \rightarrow \tau^+ + 2\eta^+$ ,  $\tau^- \rightarrow \tau^- + 2\eta^-$  is

$$\begin{aligned} T_{++} &\rightarrow T_{++} - (4\partial_+ \eta^+ T_{++} + 2\eta^+ \partial_+ T_{++}) + \frac{c}{12\pi} \partial_+^3 \eta^+ \\ T_{--} &\rightarrow T_{--} - (4\partial_- \eta^- T_{--} + 2\eta^- \partial_- T_{--}) + \frac{c}{12\pi} \partial_-^3 \eta^-. \end{aligned} \quad (6.55)$$

This is the infinitesimal version of the Schwarzian derivative that was described in Chapter 2. Because  $T_{++}$  and  $T_{--}$  were zero before the transformation (6.52), we find

$$c = \frac{3l}{2G}. \quad (6.56)$$

This central charge corresponds to the one found before in the Hamiltonian formalism if  $T_{++} = \frac{i}{\pi} \sum_{m \in \mathbb{Z}} L_m (\tau^+)^{-m-2}$ , as can be checked by means of the transformation rule given in section 2.3.

## Chapter 7

# Conformal Field Theory on the Boundary

Chapter 6 has revealed a possible analogy between boundary fluctuations in  $\text{AdS}_3$  and a conformal field theory on the  $(1+1)$ -dimensional boundary. In this chapter we wish to investigate this analogy further.

Section 7.1 first gives a short introduction to the AdS/CFT correspondence. The original AdS/CFT correspondence concerns an analogy between string theory in a background of  $\text{AdS}_d$  times some compact manifold, and supersymmetric nonabelian gauge theory in  $d-1$  dimensions. In that sense the analogy between fluctuations in  $\text{AdS}_3$  and a conformal field theory on the boundary is not a full-fledged example of an AdS/CFT correspondence. However, we can use the term in a more obvious sense.

As mentioned before, Coussaert, Henneaux, and van Driel showed in [3] that the conformal field theory living on the boundary of asymptotically  $\text{AdS}_3$  is Liouville theory. We will give a summary of their derivation. To this end we shall need Chern-Simons theory, which is the subject of section 7.2. The derivation of [3] is subsequently treated in 7.3.

### 7.1 AdS/CFT Correspondence

A correspondence between string theory in a  $d$ -dimensional anti-de Sitter background (times some compact manifold) and a gauge theory in  $d-1$  dimensions was first noted by Maldacena in [7]. The original example concerned the analogy between the near-horizon limit of  $N$  parallel D3-branes in type IIB supergravity, of which the near-horizon geometry is  $\text{AdS}_5 \times \text{S}^5$ , and another limit of four-dimensional  $\mathcal{N} = 4$  super Yang-Mills theory (i.e., nonabelian gauge theory with four independent supersymmetries). This theory has a  $U(N)$  gauge symmetry, and the correspondence holds in the limit where both  $N$  and  $g_{YM}^2 N$  are large, with  $g_{YM}$  the Yang-Mills coupling. The symmetries of the supergravity theory are enhanced to the superconformal group near the horizon, similar to the enhancement of the  $\text{AdS}_3$  symmetries to the conformal group when approaching the boundary. A first test of the correspondence is that the four-dimensional super Yang-Mills theory has the same superconformal symmetry.



Similarly, Maldacena showed that there is a correspondence between branes in type IIB string theory compactified on  $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$  and a  $(1+1)$ -dimensional conformal field theory. He then conjectured that these analogies continue to hold when moving away from the horizon, which corresponds to moving away from the conformal point on the gauge theory side. The term AdS/CFT correspondence is now more generally used for dualities between gravity and gauge theories. In this latter sense the correspondence between the Brown-Henneaux boundary fluctuations and  $(1+1)$ -dimensional Liouville theory may be viewed as an example of an AdS/CFT correspondence.

The AdS/CFT correspondence is intimately tied to the idea of holography, which says that the information contained in a  $d$ -dimensional theory can sometimes be fully captured in a  $(d-1)$ -dimensional local field theory. The term holography was coined by G. 't Hooft in 1993 [30], and he had several reasons to conjecture such a correspondence. For one, the fact that the entropy of black holes is proportional to their horizon area suggests that the degrees of freedom of gravity are nonlocal. Another indication that the degrees of freedom of general relativity are not fundamentally  $d$ -dimensional, is that large volumes with constant energy density collapse into black holes, while they are locally equivalent to small volumes with constant energy density.

For the standard AdS/CFT correspondence there is another important idea, which is that  $U(N)$  gauge theories are equivalent to string theories in the limit of large  $N$  (not to be confused with the number of supersymmetries  $\mathcal{N}$ ). This correspondence was also first suggested by 't Hooft [31]. The relevance of this result is that Yang-Mills theory is obtained in the large  $N$  limit of a configuration of  $N$  parallel D3-branes in type IIB string theory on  $\text{AdS}_5 \times \text{S}^5$  that are brought together (the length of the open strings extending between the branes is taken to zero). Maldacena's conjecture is that the full string theory that is obtained when moving away from the black hole horizon corresponds to the theory obtained when the branes are taken back to finite distance.

We will now give a more concrete description of the AdS/CFT correspondence, and see that local quantities in supergravity correspond to super Yang-Mills operators 'on the boundary' [32, 33].

A metric for  $\text{AdS}_5$  (the five-dimensional version of (5.7)) is

$$ds^2 = l^2(dr^2 + e^{2r}(\eta_{\mu\nu}dx^\mu dx^\nu)), \quad (7.1)$$

where  $\eta_{\mu\nu}$  is the four-dimensional Minkowski metric. The boundary is located at  $r = \infty$ . If  $\phi_i$  is a free field with mass  $m$  propagating in anti-de Sitter space, with equation of motion

$$(\partial^2 + m^2)\phi = 0, \quad (7.2)$$

there are two linearly independent solutions that are proportional to

$$e^{-\Delta r}, \quad e^{(\Delta-4)r} \quad (7.3)$$

near the boundary. Taking the solution that looks near the boundary like

$$\phi_i \sim \phi_i^0 e^{(\Delta-4)r}, \quad (7.4)$$

where

$$\Delta(\Delta - 4) = m^2, \quad (7.5)$$

the duality is expressed in the weak-coupling limit of string theory by

$$\exp(-S_{\text{sugra}}(\phi_i)) = \left\langle \exp \left( \int \phi_i^0 \mathcal{O}_i \right) \right\rangle. \quad (7.6)$$

Here, the supergravity action is evaluated on the classical solution  $\phi_i$ , and  $\mathcal{O}_i$  represents some operator in super Yang-Mills theory with conformal dimension  $\Delta$ . The right-hand side is the generating functional for expectation values in super Yang-Mills theory.

The AdS/CFT correspondence is a weak/strong coupling duality. On the string theory side there is a dimensionless coupling constant  $g_s$ , and two parameters with the dimension of length: the curvature radius  $l$  of  $\text{AdS}_5$  and  $S^5$ , and the string length  $l_s$ . These are related to the Yang-Mills parameters by

$$g_s = g_{YM}^2, \quad (l/l_s)^4 = 4\pi g_{YM}^2 N \equiv 4\pi\lambda. \quad (7.7)$$

The perturbative expansion for Yang-Mills theory,

$$Z = \sum_{g \geq 0} N^{2-2g} f_g(\lambda), \quad (7.8)$$

is valid in the regime where  $g_{YM}^2 N$  and  $g_{YM}$  are both small, while the expansion for string theory,

$$Z = \sum_{g \geq 0} g_s^{2g-2} Z_g, \quad (7.9)$$

is valid when  $g_{YM}$  is small and  $g_{YM}^2 N$  is large [34]. Testing the correspondence is complicated by the fact that the strong-coupling regime of neither string theory nor super Yang-Mills has been completely solved. However, the massless spectrum provides a good testing ground, and the correspondence has proven to apply at least for massless fields [35]. The reason that the massless spectrum can be solved is that the massless fields of string theory form shortened, so-called BPS multiplets, of which the conformal dimension is not renormalized. The eigenvalues of the corresponding operators on the gauge theory side are then also not renormalized (they are independent of the coupling). There is of course much more to be said about the AdS/CFT correspondence, but we shall conclude our discussion here and further refer to the literature.

## 7.2 Chern-Simons Theory

In the following, we shall use the Chern-Simons formulation of (2+1)-dimensional gravity with negative cosmological constant. Chern-Simons theory is a topological gauge theory. Therefore, recasting general relativity as a Chern-Simons theory is only possible in three dimensions, where the theory without matter terms has no local propagating degrees of freedom. The role of the gauge group is then played by the space-time isometry group. In general, a Chern-Simons

theory exists in any odd dimension, where it provides a way to describe gauge-invariant mass terms, but not in even dimensions.

The Chern-Simons Lagrangian in three dimensions for a general compact gauge group is [36]:

$$\mathcal{L}_{CS} = \kappa \epsilon^{\mu\nu\rho} \text{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) - A_\mu J^\mu, \quad (7.10)$$

where the gauge field  $A_\mu$  is Lie-algebra valued,

$$A_\mu = A_\mu^a T_a \quad (7.11)$$

with  $T_a$  the group generators, and  $\text{Tr}$  stands for a summation over the internal indices. If the group is abelian, the fields  $A_\mu$  commute, and the second term drops out because of the antisymmetry of the Levi-Civita symbol  $\epsilon^{\mu\nu\rho}$ . The equations of motion are

$$\kappa \epsilon^{\mu\nu\rho} F_{\nu\rho} = J^\mu, \quad (7.12)$$

where the field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (7.13)$$

Of course, the last term drops out in the abelian case. In Lie algebra components:

$$F_{\mu\nu}^a T_a = \partial_\mu A_\nu^a T_a - \partial_\nu A_\mu^a T_a + A_\mu^a A_\nu^b f_{ab}^c T_c, \quad (7.14)$$

with  $f_{ab}^c$  the group's structure constants,  $[T_a, T_b] = f_{ab}^c T_c$ . If the current  $J^\mu = 0$ , we obtain the source-free equations  $F_{\mu\nu} = 0$ , a demand which is equivalent to the gauge field  $A_\mu$  being pure gauge:

$$A_\mu = U^{-1} \partial_\mu U, \quad (7.15)$$

with  $U$  an element of the gauge group.

### 7.3 Liouville Field Theory

Chern-Simons theories have been shown to reduce to so-called Wess-Zumino-Witten, or WZW [37] theories on the boundary [38]. In particular, the Chern-Simons theory describing  $(2+1)$ -dimensional gravity with negative cosmological constant reduces under certain boundary conditions to an  $SL(2, R)$  WZW model on the cylinder at spatial infinity. In [3], it was shown that the Brown-Henneaux conditions are in fact stronger and cause a further reduction to Liouville theory. We will summarize the derivation, and see the nature of the equivalence.

The action for three-dimensional Einstein gravity with negative cosmological constant  $\Lambda < 0$  is a sum of two Chern-Simons actions each with gauge group  $SL(2, R)$  [15], so that the total gauge group is the product  $SL(2, R)_L \times SL(2, R)_R \simeq SO(2, 2)$ . As before, we should add a term to the action if we are going to take into account surface terms in the variation,

$$S[A, \tilde{A}] = S_{CS}[A] - S_{CS}[\tilde{A}] - \lim_{r \rightarrow \infty} \int dt d\phi \text{Tr}(A_\phi^2) - \lim_{r \rightarrow \infty} \int dt d\phi \text{Tr}(\tilde{A}_\phi^2), \quad (7.16)$$

where

$$S_{CS}[A] = \int dt dr d\phi \text{Tr}(\dot{A}_r A_\phi - \dot{A}_\phi A_r - A_0 F_{r\phi}). \quad (7.17)$$

The gauge fields  $A_\mu$ ,  $\tilde{A}_\mu$  decompose as

$$A_\mu = \omega_\mu + \frac{1}{l} e_\mu, \quad \tilde{A}_\mu = \omega_\mu - \frac{1}{l} e_\mu. \quad (7.18)$$

Writing out the group index,  $e^a_\mu$  is the dreibein field ( $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$  with  $\eta_{ab}$  the flat Minkowski metric), and  $\omega^a_\mu$  is the spin connection, which is related to the Riemann tensor by

$$\begin{aligned} R_{\mu\nu}{}^a{}_b &= R_{\mu\nu}{}^\rho{}_\sigma e^a_\rho e_b^\sigma \\ &= \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + [\omega_\mu, \omega_\nu]{}^a{}_b. \end{aligned} \quad (7.19)$$

As in Maxwell theory, the components  $A_0$  and  $\tilde{A}_0$  act as Lagrange multipliers, leading to the constraints  $F_{r\phi} = \tilde{F}_{r\phi} = 0$ . As mentioned, this amounts to the gauge field being pure gauge. The equations can thus be solved according to<sup>1</sup>

$$A_i = G_1^{-1} \partial_i G_1, \quad \tilde{A}_i = G_2^{-1} \partial_i G_2, \quad (7.20)$$

where  $i = (r, \phi)$ , and  $G_1$  and  $G_2$  are asymptotically given by

$$G_1 \sim g_1(t, \phi) \text{diag} \left( \sqrt{r}, \frac{1}{\sqrt{r}} \right), \quad G_2 \sim g_2(t, \phi) \text{diag} \left( \frac{1}{\sqrt{r}}, \sqrt{r} \right). \quad (7.21)$$

Here,  $g_1(t, \phi)$  and  $g_2(t, \phi)$  are two arbitrary  $SL(2, R)$  group elements. The particular form (7.20) for the gauge field causes a reduction of the action (7.16) to the difference of two chiral WZW actions

$$\begin{aligned} S_+^{WZW}[g_1] &= \lim_{r \rightarrow \infty} \int d\phi dt \text{Tr}(\dot{g}_1 \partial_\phi g_1 - (\partial_\phi g_1)^2) + \int dr d\phi dt \text{Tr}(g_1^{-1} dg_1)^3 \\ S_-^{WZW}[g_2] &= \lim_{r \rightarrow \infty} \int d\phi dt \text{Tr}(\dot{g}_2 \partial_\phi g_2 - (\partial_\phi g_2)^2) + \int dr d\phi dt \text{Tr}(g_2^{-1} dg_2)^3 \end{aligned} \quad (7.22)$$

The combined result turns out to be non-chiral after substituting  $g \equiv g_1^{-1} g_2$ . Besides (7.20), there are additional conditions following from the Brown-Henneaux asymptotics, which read in terms of  $A$  and  $\tilde{A}$ ,

$$\begin{aligned} A &\sim \begin{bmatrix} \frac{dr}{2r} & O\left(\frac{1}{r}\right) \\ r dx^+ & -\frac{dr}{2r} \end{bmatrix} \\ \tilde{A} &\sim \begin{bmatrix} -\frac{dr}{2r} & r dx^- \\ O\left(\frac{1}{r}\right) & \frac{dr}{2r} \end{bmatrix}. \end{aligned}$$

Translating these into conditions on  $g$  yields

$$(g^{-1} \partial_- g)^{(+)} = 1, \quad (\partial_+ g g^{-1})^{(-)} = 1. \quad (7.23)$$

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<sup>1</sup>The derivation is slightly modified for black hole solutions [3].

In order to incorporate these extra conditions, the authors of [3] Gauss decomposed

$$g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{1}{2}\phi) & 0 \\ 0 & \exp(-\frac{1}{2}\phi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \quad (7.24)$$

obtaining the action

$$S^{WZW} = \int dt d\phi [\frac{1}{2} \partial_+ \phi \partial_- \phi + 2(\partial_- X)(\partial_+ Y) \exp(-\phi)]. \quad (7.25)$$

There is one last complication arising from the fact the constraints (7.23) restrict  $\partial_- Y$  and  $\partial_+ X$  rather than  $X$  and  $Y$  themselves. The latter would be appropriate for (7.25), but since the former applies, we need to add another boundary term to the action:

$$S_{impr}^{WZW} = \int dt d\phi [\frac{1}{2} \partial_+ \phi \partial_- \phi + 2(\partial_- X)(\partial_+ Y) e^{-\phi}] - 2 \oint d\phi (X \partial_+ Y + Y \partial_- X) e^{-\phi} \Big|_{t_1}^{t_2}. \quad (7.26)$$

Substituting the additional constraints (7.23) finally leads to the Liouville action

$$S[A, \tilde{A}] = \int dt d\phi (\frac{1}{2} \partial_+ \phi \partial_- \phi + 2e^\phi). \quad (7.27)$$

This time, the action does not contain the boundary term that contributed to the central charge of the algebra of Noether charges in the example of section 2.4. Therefore, the analogy has not preserved the central charge of the Virasoro algebra. In fact, the Liouville theory (7.27) has an effective central charge  $c_{eff} = 1$ . In [39], it was proposed that the central charge  $c = \frac{3l}{2G}$  is obtained when counting all modes, including those that are non-normalizable, whereas  $c = 1$  counts only normalizable modes. The relation between the Liouville field  $\phi$  and the original variables  $e_\mu^a$  and  $\omega_\mu^a$  is rather complex, but may be traced back through the above derivation.

It would be interesting to know if analogies such as the one described apply to a wider range of theories. However, a generalization to higher dimensions or theories with matter terms is not straightforward, since in these cases the action can no longer be rewritten as a Chern-Simons action. Therefore, it remains to be seen whether this result can find a wider application.

## Chapter 8

# Entropy

We conclude this thesis with a note on black hole entropy. There have been successful calculations of black hole entropy in string theory for extreme and near-extreme black holes [40]. Interestingly, Strominger has shown [9] that we can use the *classical* central charge of asymptotically  $\text{AdS}_3$  to calculate the entropy of the BTZ black hole. The calculation relies on the Cardy formula [8] for the asymptotic density of states, which is a well-known result from conformal field theory. The Cardy formula is specific to two dimensions. It therefore seems that a generalization to other dimensions is only possible if the relevant solution has a special two-dimensional boundary. Although the formula works well for the BTZ black hole, other examples have been found where the correspondence does not hold exactly. A tentative solution to this problem is that Cardy's formula only gives a maximum possible entropy for a given mass [41].

There is a well-known analogy between black hole physics and thermodynamics, called black hole thermodynamics. In this analogy, the role of temperature is played by the surface gravity  $\kappa$ ,

$$T_{bh} = \frac{\kappa}{2\pi}. \quad (8.1)$$

Under reasonable circumstances, the horizon of a stationary black hole is a Killing horizon, where Killing vectors become null. If  $\chi^\mu$  is a normal Killing vector field to the horizon,  $\kappa$  is defined by

$$D^\nu(\chi^\mu \chi_\mu) = -2\kappa \chi^\nu. \quad (8.2)$$

The entropy, on the other hand, is proportional to the horizon area<sup>1</sup>,

$$S_{bh} = \frac{A}{4G}, \quad (8.3)$$

This is the Bekenstein-Hawking entropy [10, 11]. The analogy suggests that there should be a statistical mechanical interpretation explicitly counting the microstates of the black hole. In  $2 + 1$  dimensions there turns out to be an

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<sup>1</sup>This is how the Bekenstein-Hawking entropy is usually written. Bringing back the appropriate constants, it is actually  $S_{bh} = \frac{Akc^3}{4hG}$ , where  $k$  is Boltzmann's constant,  $c$  the speed of light, and  $h$  Planck's constant.

elegant way to do this, using the central charge of the Virasoro algebra on the boundary. This was first suggested by Strominger in [9]. The method makes use of Cardy's formula for the asymptotic (i.e., large  $\Delta$  and  $\bar{\Delta}$ ) density of states of a conformal field theory with central charge  $c$ ,

$$\rho(\Delta, \bar{\Delta}) \approx \exp \left[ 2\pi \sqrt{\frac{c\Delta}{6}} + 2\pi \sqrt{\frac{\bar{c}\bar{\Delta}}{6}} \right], \quad (8.4)$$

Here,  $\Delta$  and  $\bar{\Delta}$  are the eigenvalues of  $l_0$  and  $\bar{l}_0$ , respectively,

$$\Delta = \frac{1}{2}(lM + J), \quad \bar{\Delta} = \frac{1}{2}(lM - J). \quad (8.5)$$

Large values of  $\Delta$  and  $\bar{\Delta}$  imply large mass as well as the non-extreme limit  $lM \gg J$ . The entropy can be obtained as the logarithm of the density of states. Strominger applied this to the BTZ black hole, for which the Bekenstein-Hawking entropy (8.3) becomes

$$S_{bh} = \frac{\pi \sqrt{16GMl^2 + 8Gl\sqrt{M^2l^2 - J^2}}}{4G}. \quad (8.6)$$

It can be verified that this is the same result as is obtained by substituting (8.5) and  $c = \bar{c} = \frac{3l}{2G}$  in (8.4), and the Cardy formula yields the correct density of states for the BTZ black hole. This is a remarkable result, since we have derived the central charge from fully classical considerations. Since the Cardy formula is derived in the context of conformal field theory, the derivation as a whole is semiclassical.

There have been various attempts to apply this method to more general solutions. The approach has been successful for various black hole solutions whose near-horizon geometry is that of the BTZ black hole<sup>2</sup>. In [42], the three-dimensional Martínez-Zanelli (MZ) black hole, which includes a conformal scalar field and is asymptotically anti-de Sitter, was considered, yielding a result that corresponds to the Bekenstein-Hawking entropy up to a prefactor. The discrepancy is possibly related to the fact that the solution is only one arbitrary member of a class of space-times with the same asymptotics. This makes it debatable to what extent the boundary dynamics belong to this particular solution. It was therefore suggested in [14] that the Cardy formula only gives the maximal possible entropy for solutions with mass  $M$  and the same asymptotic behavior. The BTZ black hole would then be the solution with maximum entropy, for which the Cardy formula gives the correct result.

There are many more problems with the approach of Strominger. For one, Cardy's formula is only supposed to work for BTZ black holes with large mass and relatively small angular momentum, but yields the correct entropy for any values of  $M$  and  $J$ .

Moreover, the derivation has assumed a ground state with mass  $M = 0$  and angular momentum  $J = 0$ , so that the lowest possible eigenvalues of  $l_0$  and

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<sup>2</sup>For a list of references, see [42].

$\bar{l}_0$  become  $\Delta_0 = \bar{\Delta}_0 = 0$ . If the lowest Virasoro eigenvalues are different, we instead have to use

$$c_{eff} = c - 24\Delta_0, \quad \bar{c}_{eff} = \bar{c} - 24\bar{\Delta}_0. \quad (8.7)$$

For instance, using  $\text{AdS}_3$  ( $M = -\frac{1}{8G}$ ,  $J = 0$ ) as a groundstate would yield  $c_{eff} = \bar{c}_{eff} = \frac{6l}{2G}$ , and a different value for the entropy.

In addition, it seems mysterious that the Bekenstein-Hawking entropy, which is proportional to horizon area, should be derivable by counting the degrees of freedom on the boundary at infinity. To make the argument more intuitive, Carlip [14, 43] presented a near-horizon construction valid for black holes in any dimension, but unfortunately with some seemingly arbitrary boundary conditions. Several possible explanations have been given for the fact that the derivation also works at spatial infinity, although none of them are quite satisfactory. In [44], a mechanism was suggested by which the central charge may “flow from the boundary at infinity to the horizon”. It has also been proposed that the possibility of a near-infinity calculation is due to the simplicity of three-dimensional gravity. This argument seems to disregard the fact that the first law of black hole thermodynamics relates the entropy to quantities at infinity in any dimension (see appendix). Therefore, the significance of this result remains to be investigated.

Finally, the Bekenstein-Hawking entropy is a formula derived in the context of semiclassical gravity, and may yet be subject to quantum corrections. By improving upon the approximation scheme that led to (8.4), Carlip [41] calculated a prefactor for Cardy’s formula,

$$\rho(\Delta, \bar{\Delta}) \approx \left(\frac{c}{96\Delta^3}\right)^{1/4} \left(\frac{\bar{c}}{96\bar{\Delta}^3}\right)^{1/4} \exp \left[ 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{\Delta}}{6}} \right]. \quad (8.8)$$

Various near-horizon treatments of diffeomorphisms in the  $r-t$  plane in general dimension lead to the following values for  $c$  and  $\Delta$  [41],

$$c = \frac{3A}{2\pi G} \frac{\beta}{\kappa} \quad (8.9)$$

and

$$\Delta = \frac{A}{16\pi G} \frac{\kappa}{\beta}, \quad (8.10)$$

where  $\beta$  is an undetermined periodicity. Inserting these values into the Cardy formula (8.4), this leads to the standard Bekenstein-Hawking entropy. However, inserting it into (8.8) leads to a logarithmic correction to the Bekenstein-Hawking entropy formula,

$$S \sim \frac{A}{4\hbar G} - \frac{3}{2} \ln \left( \frac{A}{4\hbar G} \right) + \dots \quad (8.11)$$

Taking all these problems into consideration, the subject remains open to many avenues of research.



## Chapter 9

# Discussion

We have reproduced the result of Brown and Henneaux [2] that the asymptotic isometry group of  $\text{AdS}_3$  is extended to the infinite-dimensional conformal group in 1+1 dimensions when using appropriate boundary conditions. This is a result specific to three dimensions, since the conformal group has only a finite number of generators in dimensions higher than two. The asymptotic isometry group of  $\text{AdS}_3$  is generated by two copies of the Virasoro algebra with central charge  $c = \frac{3l}{2G}$ . The central charge shows up at the level of the Poisson brackets and is therefore classical. We have also seen other examples of classical central charges, and shown that there are two types: those that generate a transformation and those that do not. The Brown-Henneaux central charge falls into the latter category.

The fact that the asymptotic symmetry algebra is the Virasoro algebra with central charge  $\frac{3l}{2G}$  leads to the assumption that fluctuations around  $\text{AdS}_3$  are described by a two-dimensional conformal field theory with the same central charge. We have summarized the derivation of [3], in which it was shown that this conformal field theory is Liouville theory. It seems puzzling at first that this theory has an effective central charge  $c = 1$ . However, a possible resolution has been proposed in [39].

The central charge of asymptotically  $\text{AdS}_3$  space-times has been used [9] to reproduce the Bekenstein-Hawking entropy of the BTZ black hole through Cardy's formula [8] for the asymptotic density of states. This result is remarkable in two respects. First of all, it suggests that the degrees of freedom of a black hole may not fundamentally live on the black hole horizon. Moreover, it suggests that black hole entropy has a classical origin. Unfortunately, there are some problems with the approach of [9], which we have discussed. It will be interesting to see whether these issues can be resolved without damaging the original argument.

# Appendix A

## Conserved Quantities

We give a short characterization of the relation between the surface charges and conserved quantities in general relativity. The surface charges  $J(\xi)$  can be written as the integral of an antisymmetric Noether potential  $Q^{\mu\nu}(\xi)$  over the boundary at spatial infinity,

$$J(\xi) = \int_{\Sigma_t^\infty} \sqrt{h} \, \epsilon_{\mu\nu} Q^{\mu\nu}(\xi) \, d^{d-2}x. \quad (\text{A.1})$$

$Q^{\mu\nu}(\xi)$  derives its name from the fact that, on shell, its derivative is the Noether current belonging to  $\xi$ ,

$$T^\mu(\xi) = \partial_\nu Q^{\mu\nu}(\xi), \quad (\text{A.2})$$

and  $T^\mu$  is automatically conserved ( $\partial_\mu T^\mu = 0$ ) by the antisymmetry of  $Q^{\mu\nu}$ . However, these conserved currents do not give rise to the usual Noether charges. The conserved quantities are instead obtained from integrals like (A.1), for only those vector fields  $\xi$  that are isometries. Surface charges evaluated on Killing vectors in the time and angular directions give an expression for the total mass and angular momentum of a solution, respectively. If the solution is a black hole, a third conserved quantity, the entropy, can be found by integrating the same expression over a spatial cross section of the Killing horizon,

$$S = - \int_{\Sigma_{\text{hor}}} \sqrt{h} \, \epsilon_{\mu\nu} Q^{\mu\nu}(\xi) \, d^{d-2}x \, |_{\xi^\mu=0, \nabla_{[\mu}\xi_{\nu]}=\epsilon_{\mu\nu}} \quad (\text{A.3})$$

where  $\epsilon_{\mu\nu}$  is a binormal spanned by two lightlike vectors at the horizon [45] normalized according to  $\epsilon^{\mu\nu}\epsilon_{\mu\nu} = -2$ . The integral (A.3) yields the Bekenstein-Hawking entropy  $S_{bh} = \frac{A}{4G}$  in the case of vacuum general relativity (i.e.,  $Q^{\mu\nu}$  is the Noether potential belonging to the Einstein-Hilbert action without matter terms).

The first law of black hole thermodynamics,

$$\delta M = \frac{\kappa}{2\pi} \delta S + \Omega \delta J + \Phi \delta Q + \dots \quad (\text{A.4})$$

where  $\Omega$  is the angular velocity of the horizon,  $\Phi$  the electric potential, and  $Q$  the electric charge of the black hole, thus relates integrals at infinity with a local quantity at the horizon [46]. This may have a connection with the fact that we were able to derive an expression for the entropy, which seems to live on the horizon, through an analysis conducted at infinity.

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